Characterization of minimal cycle obstruction sets for partitionable planar graphs

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Background

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How about:

- Partition graphs that belong to some more general class?
- Partition the vertex-set into "stable-like" sets?
  - Each set induces a graph with bounded maximum degree.
  - Each set induces a graph with bounded component size.

A more general class

Hadwiger's Conjecture: The vertex-set of every K_{t+1}-minor-free graph can be partitioned into t stable sets.

Theorem: Let G be a K_{t+1}-minor free graph.

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2. **(Edwards, Kang, Kim, Oum, Seymour)** For every k, there exists a K_{t+1}-minor-free graph that does not admit a partition into $t-1$ parts where each induces a graph of maximum degree less than k.

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4. **(L., Oum)** $V(G)$ can be partitioned into 3 parts where each part induces a graph of bounded component size, if G has bounded maximum degree.

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4. **(L., Oum)** $V(G)$ can be partitioned into 3 parts where each part induces a graph of bounded component size, if G has bounded maximum degree.
5. For every k, there exists a planar graph with maximum degree 6 that does not admit a partition into 2 parts where each induces a graph of diameter less than k.

Planar graphs

Theorem: Let G be a planar graph.

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3. **(Esperet, Joret)** For every k, there exists a planar graph that does not admit a partition into 3 parts each induces a graph of diameter less than k.

Planar graphs with some cycles forbidden

Let S be a set. A graph is S-free if it does not contain any subgraph isomorphic to some member of S.

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How about partitioning $V(G)$ into stable-like sets for planar graphs G with some cycles forbidden?

Improper coloring

A graph is $(k_1, k_2, ..., k_t)$-colorable if its vertex-set can be partitioned into t sets $X_1, X_2, ..., X_t$ such that $G[X_i]$ has maximum degree at most k_i for every i.

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Theorem: Let G be a planar graph.

1. **(4CT)** G is $(0, 0, 0, 0)$-colorable.
2. **(Cowen, Cowen, Woodall)** G is $(2, 2, 2)$-colorable.

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Theorem: Let G be a planar graph.

1. **(4CT)** G is $(0, 0, 0, 0)$-colorable.
2. **(Cowen, Cowen, Woodall)** G is $(2, 2, 2)$-colorable.
3. For every k, some planar graph is not $(1, k, k)$-colorable.

Improper coloring planar graphs with cycles forbidden

Theorem: Let G be a planar graph.

1. **(Grötzsch)** If G is $\{C_3\}$-free, then G is $(0, 0, 0)$-colorable.
2. If G is $\{C_3, C_4\}$-free, then G is
    - $(1, 10)$-colorable **(Choi, Choi, Jeong, Suh)**
    - $(2, 6)$-colorable **(Borodin, Kostochka)**
    - $(3, 5)$-colorable **(Choi, Raspaud)**
    - $(4, 4)$-colorable **(Škrekovski)**

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What is the minimal set of cycles such that excluding those cycles ensures the existence of an improper coloring?

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A graph is $(0, *)$-colorable if there exists M such that it is $(0, M)$-colorable.

Theorem (Choi, L., Oum)

1. *The minimal* $(*, *)$*-obstruction sets are* $\{C_4\}$ *and the set of all odd cycles.*
2. *The minimal* $(0, *)$*-obstruction sets are* $\{C_3, C_4, C_6\}$ *and the set of all odd cycles.*
3. *The minimal* $(0, 0, *)$*-obstruction sets are* $\{C_3\}$ *and* $\{C_4\}$*.*

$(0,*)$-obstruction

Lemma

For every positive integer k and ℓ, there exist non-$(0,k)$-colorable planar graphs G_1, G_2, G_3, G_4 such that

- *every cycle in G_1 has length 4 or $2\ell+1$,*
- *every cycle in G_2 has length 3,*
- *every cycle in G_3 has length 6 or $4\ell+1$, and*
- *every cycle in G_4 has length 6 or $4\ell+3$.*

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- *every cycle in G_4 has length 6 or $4\ell + 3$.*

Lemma

If S is a $(0, *)$-obstruction set, then either S contains all odd cycles, or S contains $\{C_3, C_4, C_6\}$.

$(0, *)$-obstruction

Lemma

Let G be an S-free planar graph.

1. *If S contains all odd cycle, then G is $(0, 0)$-colorable.*
2. *If $S = \{C_3, C_4, C_6\}$, then G is $(0, 45)$-colorable.*

$(0, *)$-obstruction

Lemma

Let G be an S-free planar graph.

1. *If S contains all odd cycle, then G is $(0, 0)$-colorable.*
2. *If $S = \{C_3, C_4, C_6\}$, then G is $(0, 45)$-colorable.*

Theorem

The minimal $(0, *)$-obstruction sets are $\{C_3, C_4, C_6\}$ and the set of all odd cycles.

Questions

- Minimal obstruction sets for partitioning planar graphs into (2 or 3) graphs with bounded component size?
- Minimal obstruction sets for partitioning more general graphs into graphs with bounded maximum degree?
- Is it possible to partition every K_{t+1}-minor-free graph with no triangle into less than t graphs with bounded maximum degree/component size?
- Minimal obstruction sets for partitioning graphs into sparse graphs?

THANK YOU!