Spectral problems:
Numerical analysis, validation, and proof.

Nilima Nigam
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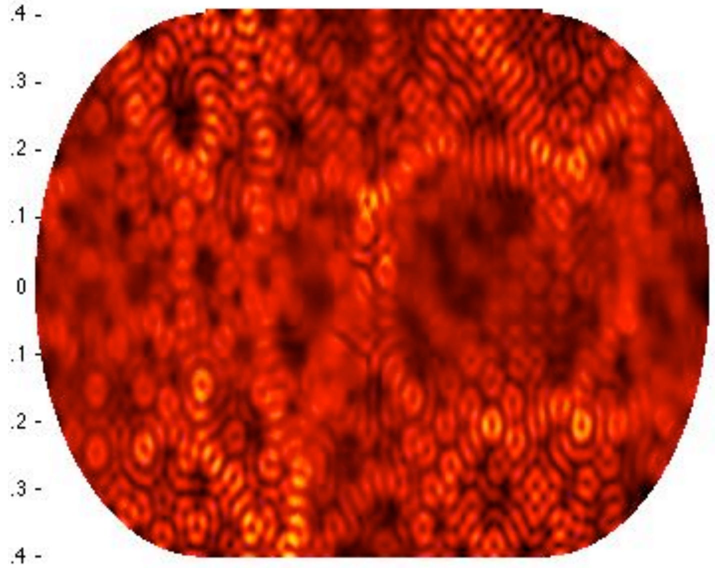

July 31, 2013

First disclaimer

First disclaimer

From: D. N. Arnold, http://www.ima.umn.edu/~arnold/disasters/sleipner.html

The sinking of the Sleipner A offshore platform

Excerpted from a report of SINTEF, Civil and Environmental Engineering:

The Sleipner A platform produces oil and gas in the North Sea and is supported on the seabed at a water depth of 82 m. It is a Condeep type platform with a concrete gravity base structure consisting of 24 cells and with a total base area of 16 000 m^2. Four cells are elongated to shafts supporting the platform deck. The first concrete base structure for Sleipner A sprang a leak and sank under a controlled ballasting operation during preparation for deck mating in Gandsfjorden outside Stavanger, Norway on 23 August 1991.

Immediately after the accident, the owner of the platform, Statoil, a Norwegian oil company appointed an investigation group, and SINTEF was contracted to be the technical advisor for this group.

The investigation into the accident is described in 16 reports...

The conclusion of the investigation was that the loss was caused by a failure in a cell wall, resulting in a serious crack and a leakage that the pumps were not able to cope with. The wall failed as a result of a combination of a serious error in the finite element analysis and insufficient anchorage of the reinforcement in a critical zone.

A better idea of what was involved can be obtained from this photo and sketch of the platform. The top deck weighs 57,000 tons, and provides accommodation for about 200 people and support for drilling equipment weighing about 40,000 tons. When the first model sank in August 1991, the crash caused a seismic event registering 3.0 on the Richter scale, and left nothing but a pile of debris at 220m of depth. The failure involved a total economic loss of about $700 million.

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Eigenproblems. *SIAM J. Sci. Comput.* 35 (2013), no. 3, A1689–A1708.
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MR3047040 Prelim Jia, ShangHui; Chen, HongTao; Xie, HeHu; A posteriori error estimator for eigenvalue problems by m element method. *Sci. China Math.* 56 (2013), no. 5, 887–900.
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Proof and numerical computations

Goal: Formulate some conjecture about eigenpair (u, λ).

- Formulate conjecture based on numerically computed approximations to (u, λ).

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- Proof strategy includes numerically computed approximations to (u, λ).

How and when are such approximations acceptable in conjectures and proofs?

Introduction

Abstract setting

Spectral approximation

- In Banach spaces
- Numerical linear algebra

Examples

- Spectral problem 1
- Spectral problem 2

Let's examine the process of computing eigenpairs (λ, u)

Infinite dim.
function space

Finite dim.
function space

Finite dim.
Eucl. space

Finite precision
computation

Let's examine the process of computing eigenpairs (λ, u)

Infinite dim. function space	Space $\mathcal{H}$ Operator $\mathcal{A}$
Finite dim. function space	Space H_N Operator $\mathcal{A}_N$
Finite dim. Eucl. space	Space $\mathbb{C}_N$ $\mathtt{A}_{NM}, \mathtt{B}_{NM}$
Finite precision computation	Space $\mathbb{C}_N$ $\tilde{\mathtt{A}}_{NM}, \tilde{\mathtt{B}}_{NM}$

Let's examine the process of computing eigenpairs (λ, u)

Infinite dim. function space	Space $\mathcal{H}$ Operator $\mathcal{A}$	Want $(u, \lambda) \in (\mathcal{H}, \mathbb{C})$
Finite dim. function space	Space H_N Operator $\mathcal{A}_N$	
Finite dim. Eucl. space	Space $\mathbb{C}_N$ $\mathtt{A}_{NM}, \mathtt{B}_{NM}$	
Finite precision computation	Space $\mathbb{C}_N$ $\tilde{\mathtt{A}}_{NM}, \tilde{\mathtt{B}}_{NM}$	Compute $(\tilde{\mathtt{u}}_N, \tilde{\ell}) \in (\mathbb{C}^N, \mathbb{C})$

Process contd.

Problem setting		Find:
Infinite dim. function space	$\mathcal{A} : \mathrm{dom}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$	$(u, \lambda) \in (\mathcal{H}, \mathbb{C})$ $\mathcal{A}u = \lambda u$
Finite dim. function space	$\mathcal{A}_N : H_N \to H_N$	$(u_N, \lambda_N) \in (H_N, \mathbb{C})$ $\mathcal{A}_N u_N = \lambda_N u_N$
Finite dim. Eucl. space	$\mathtt{A}_{NM}, \mathtt{B}_{NM} : \mathbb{C}^N \to \mathbb{C}^M$	$(\mathtt{u}_N, \Lambda_N) \in (\mathbb{C}^N, \mathbb{C})$ $\mathtt{A}_{NM}\mathtt{u}_N = \mathtt{B}_{NM}(\Lambda_N)\mathtt{u}_N$
Finite computation	$\tilde{\mathtt{A}}_{NM}, \tilde{\mathtt{B}}_{NM} : \mathbb{C}^N \to \mathbb{C}^M$	$(\tilde{\mathtt{u}}_N, \tilde{\Lambda}_N) \in (\mathbb{C}^N, \mathbb{C})$ $\tilde{\ell}_N \approx \tilde{\Lambda}_N, \tilde{\mathtt{w}}_N \approx \tilde{\mathtt{u}}_N.$

Goal: Prove some conjecture about (u, λ).

- Obtain, numerically, $\tilde{\ell} \in \mathbb{C}, \tilde{\mathtt{w}}_N \in \mathbb{C}^N$.

- Is $\tilde{\ell}$ close to λ?
- Is $\tilde{\mathtt{w}}_N$ "close" to $u \in \mathcal{H}$?
- Is their use in a proof acceptable?

First choice: formulation

How to write spectral problem as $\mathcal{A}u = \lambda u$?
eg., how to find eigenvalues of the Laplacian in a bounded domain?

- Formulation using differential operator.

$$\Delta u = -\lambda u$$

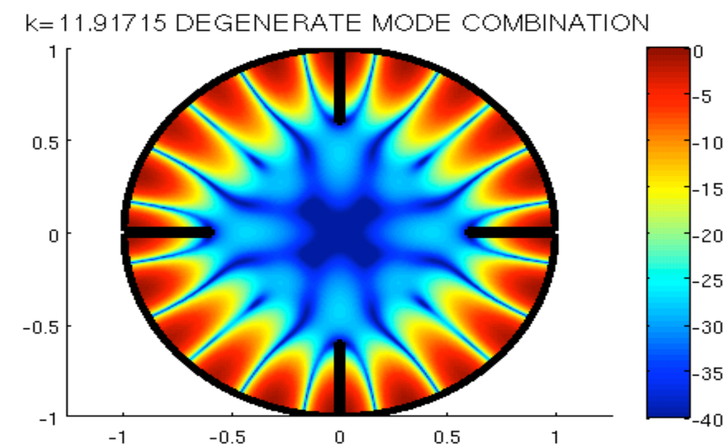
- Formulation using integral operators. [Steinbach '10, Akhmetgaliyev, Bruno and NN '13]

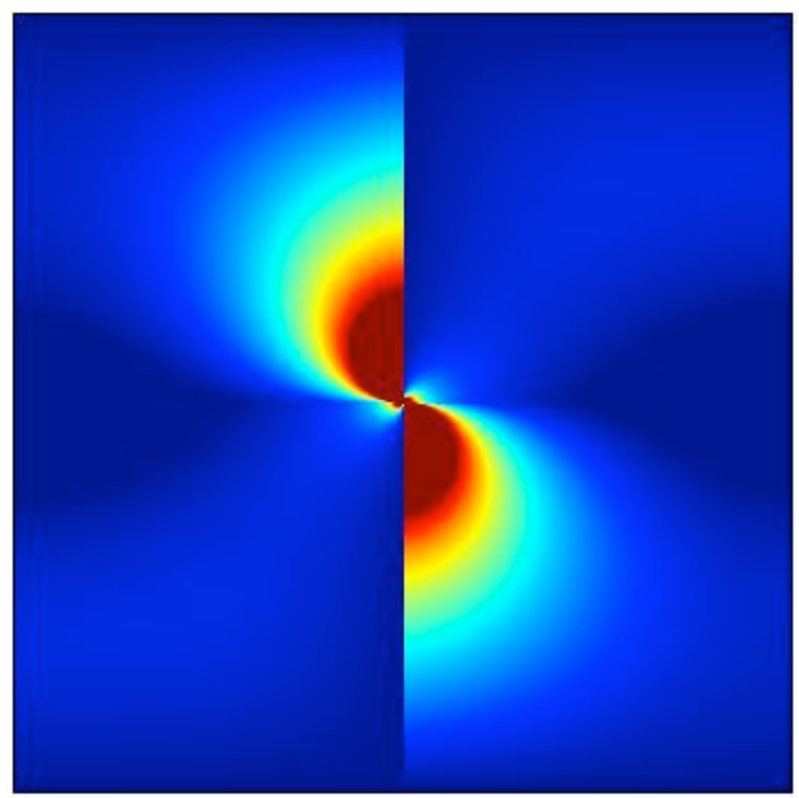
$$u_\lambda = \text{layer potential}_\lambda \phi$$

- Formulation in weaker setting using mixed methods. [Boffi, 2010]

$$\sigma = \nabla u, \quad \nabla \cdot \sigma = \lambda u.$$

We shall focus on formulations in terms of differential operators.





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Approximation in Banach spaces

Infinite dim. function space	$\mathcal{A} : \mathrm{dom}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$
Finite dim. function space	$\mathcal{A}_N : H_N \to H_N$

Approximation in Banach spaces

Infinite dim. function space

$$\mathcal{A} : \mathrm{dom}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$$

Finite dim. function space

$$\mathcal{A}_N : H_N \to H_N$$

- $\mathcal{H}$ a separable Banach space
- $\mathcal{A}$ a closed linear operator
- H_N a finite-dimensional Banach space of dimension N
- $r_N : \mathcal{H} \to H_N$ a restriction map.

[Chatelin, 1973.]

Want some notion of convergence

Let $\mathcal{A} : \mathrm{dom}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$, and λ be an isolated eigenvalue of multiplicity m. Let $\mathcal{A}_N : H_N \to H_N$.

Spectral projection

Let Γ be a closed Jordan curve around λ, $D_\lambda := Int(\Gamma)$. Define the spectral projection of $\mathcal{A}$ by

$$\mathcal{S} := -\frac{1}{2\pi\iota}\int_\Gamma (\mathcal{A} - z\mathcal{I})^{-1}\,dz, \qquad dim(\mathcal{S}\mathcal{H}) = m.$$

Define the spectral projection of $\mathcal{A}_N$ by

$$\mathcal{S}_N := -\frac{1}{2\pi\iota}\int_\Gamma (\mathcal{A}_N - z\mathcal{I})^{-1}\,dz,$$

the spectral projection associated with all eigenvalues of $\mathcal{A}_N$ in D_λ.

Convergence of spectral approximations

Definition: Convergence of approximations

The spectral element $(\lambda_N, \mathcal{S}_N)$ of $\mathcal{A}_N$ converges to the spectral element $(\lambda, \mathcal{S})$ of $\mathcal{A}$ iff

- Given $\epsilon > 0$, $\sigma(\mathcal{A}_N) \cap B_\epsilon(\lambda) \neq \{\phi\}, \forall N$ large enough;
- $\lim_{N\to\infty}\{\sigma(\mathcal{A}_N) \cap D_\lambda\} = \{\lambda\}$
- $\|(r_N\mathcal{S} - \mathcal{S}_N r_N)w\|_N \to 0$, for all $w \in \mathcal{H}$.

[Chatelin, 1973]

Approximation in Banach spaces

What do we need for convergence?

Approximation in Banach spaces

What do we need for convergence?

$$\mathcal{A}u = \lambda u, \quad u \in \mathcal{H} \qquad \text{and} \quad \mathcal{A}_N u_N = \lambda_N u_N, \quad u_N \in H_N$$

Examine action of $\mathcal{A}_N$ on $w \in \mathrm{dom}(\mathcal{A})$.

$$r_N w \in H_N, \Rightarrow \mathcal{A}_N(r_N w) \in H_N \quad \text{and} \quad \mathcal{A}w \in \mathcal{H}, \Rightarrow r_N \mathcal{A}w \in H_N$$

Approximation in Banach spaces

What do we need for convergence?

$$\mathcal{A}u = \lambda u, \quad u \in \mathcal{H} \qquad \text{and} \quad \mathcal{A}_N u_N = \lambda_N u_N, \quad u_N \in H_N$$

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Definition: Consistency of approximation.

The approximation using restriction map r_N, and operator $\mathcal{A}_N$ is consistent iff for all $w \in \mathrm{dom}(\mathcal{A})$,

$$\|r_N w\|_{H_N} \to \|w\|_{\mathcal{H}}, \ \forall w \in \mathcal{H}, \qquad \|(\mathcal{A}_N r_N - r_N \mathcal{A})w\|_{H_N} \to 0,$$

Is this enough to guarantee convergence?

Chatelin's example

- Banach space $X := \{u \in C[0,1], u(0) = u(1)\}$ with the max norm.
- Operator $T : dom(T) \to X$ is $T := \frac{d}{dx}$
- Simple eigenvalues, integer multiples of $2\pi\iota$.

Chatelin's example

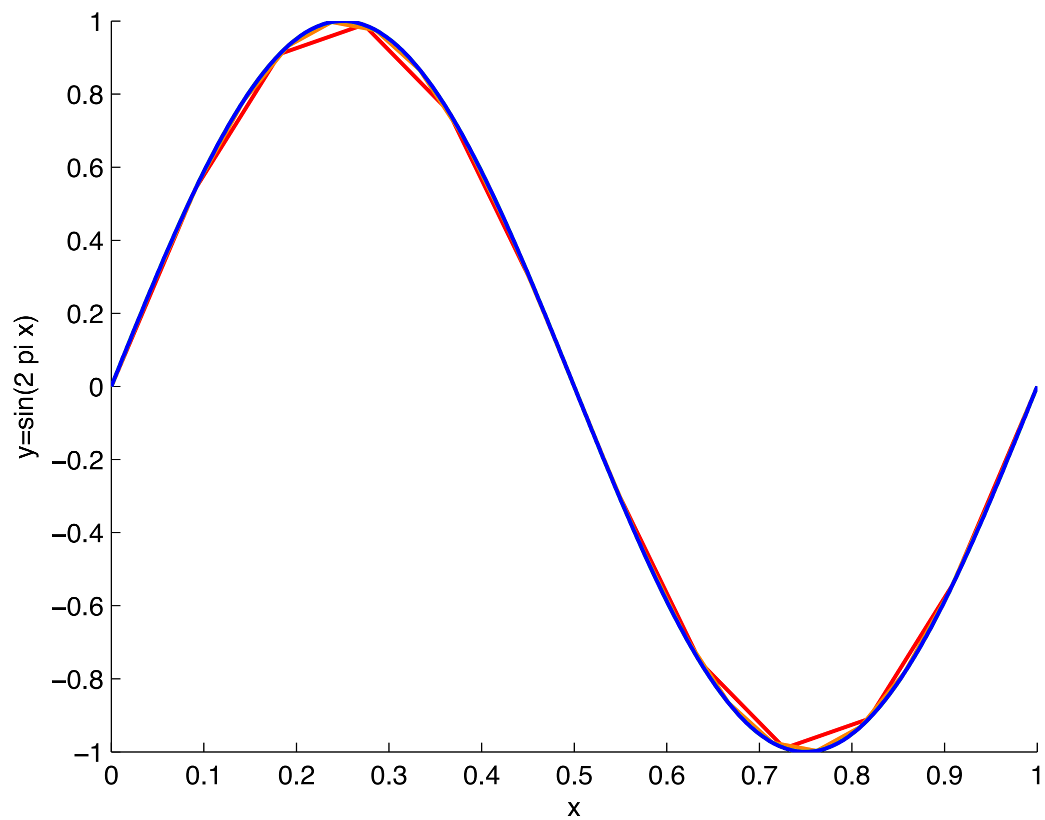
- Banach space $X := \{u \in C[0,1], u(0) = u(1)\}$ with the max norm.
- Operator $T : dom(T) \to X$ is $T := \frac{d}{dx}$
- Simple eigenvalues, integer multiples of $2\pi\iota$.
- Finite dimensional space $X_N = \mathbb{C}^N$, with $h = \frac{1}{N}$
- Restriction map $r_N : x(t) \to x(t_i), t_i = ih$
- Discrete operator T_N corresponding to first-order finite difference approximation,

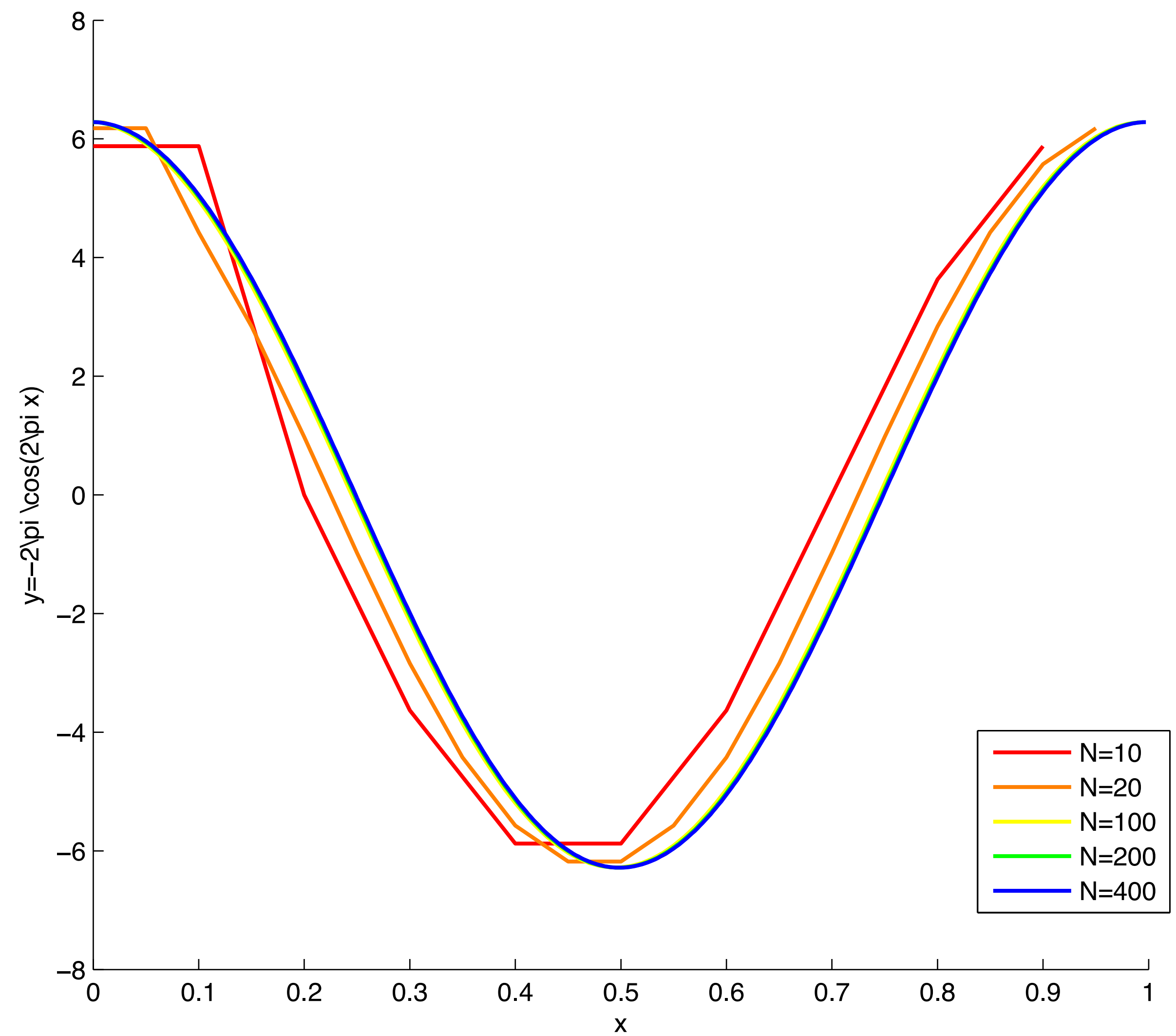
$$\frac{d}{dx}u(x_i) \approx \frac{u(x_{i+1}) - u(x_i)}{x_{i+1} - x_i}$$

- Do reasonable things at boundary: $x_0 = x_n$; $\frac{x_N - x_{N-1}}{h} = \lambda x_{n-1}$
- λ_N computed for discrete operator T_N

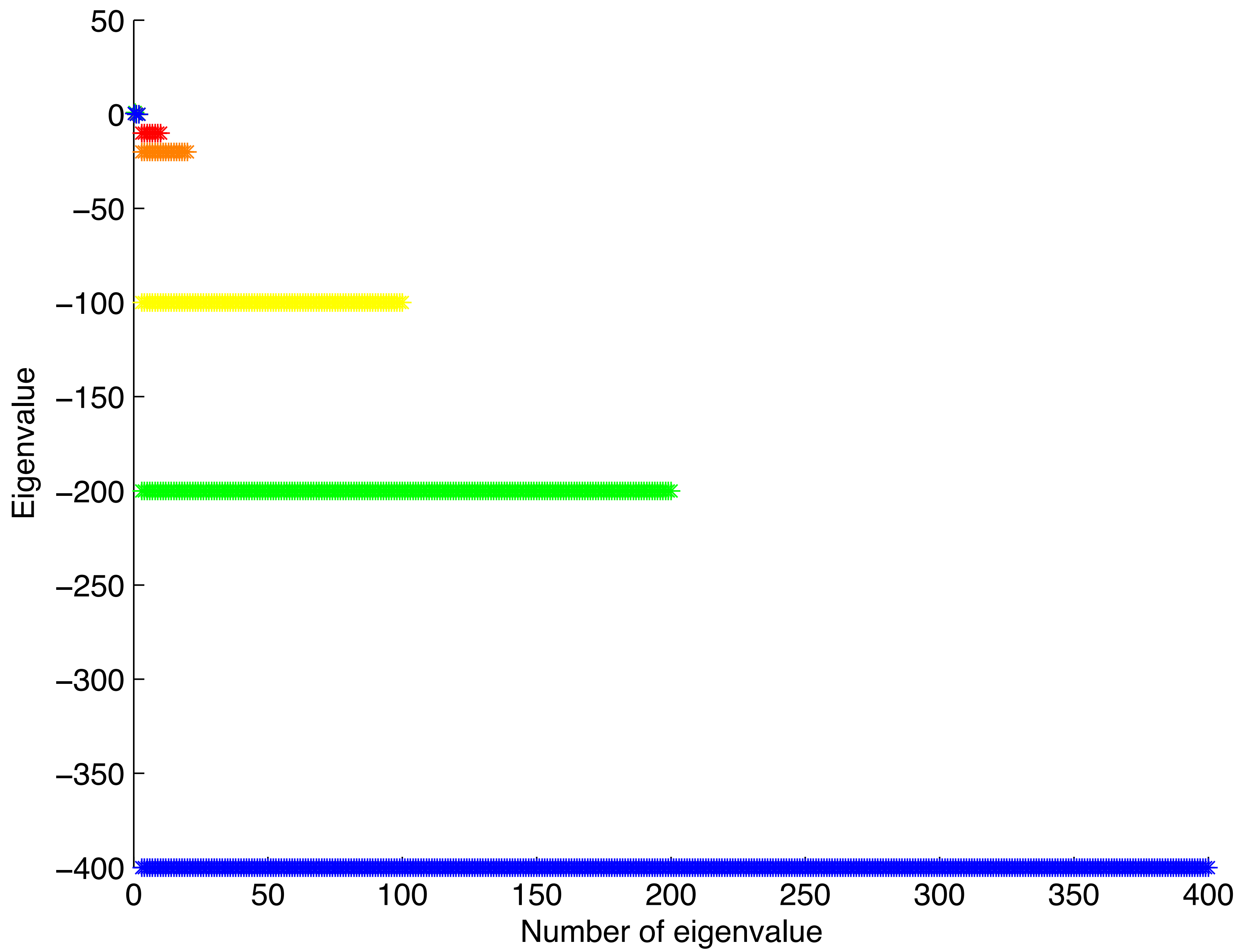
Chatelin's example. True eigenvalues are $2k\pi\iota$.

T_h seems to approximate T well. For example,
$T_h r_h \cos(2\pi x) \to T\cos(2\pi x) = -2\pi sin(2\pi x)$





Chatelin's example. $\lambda_{true} = 2k\pi\iota, \lambda_h = 0, -\frac{1}{h}$



Key ingredient is stability

Strong stability

$\mathcal{A}_N$ is strongly stable in D_λ iff

- $\mathcal{A}_N$ is stable: Given $M > 0$, $z \in D_\lambda, z \neq \lambda$, for all N large enough $\|(\mathcal{A}_N - z\mathcal{I})^{-1}\|_N \leq M$;
- r_N is linear and stable; and
- $\mathrm{dim} \mathcal{S}_N H_N = \mathrm{dim} \mathcal{S}\mathcal{H}$

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- r_N is linear and stable; and
- $\mathrm{dim} \mathcal{S}_N H_N = \mathrm{dim} \mathcal{S}\mathcal{H}$

Meta theorem

Consistency + Stability $\Rightarrow$ Convergence

Projection methods

- $\mathcal{H}$ a separable Banach space, $\mathcal{A}$ a closed operator, H_N a finite-dimensional Banach space of dimension N

- $r_N = P_N : \mathcal{H} \to H_N$ a projection map.

[Kantorovitch, 1948, Anselone 1971]

eg: Lagrange interpolation

- $\mathcal{H} := C[-1, 1]$ with sup norm

- $P_N : \mathcal{H} \to$ polynomials of degree $\leq N$, with $P_N u :=$ Lagrange interpolant for $\{t_i\}_{i=0}^N$.

- $\|P_N\|_\infty \geq \frac{2}{\pi^2} \log(N-1) + b(N)$.

- P_N is bounded as a map into $L^2_w[-1, 1]$ with weight $(1-t^2)^{-1/2}$.

Second choice: approximation space H_N

- Choose $H_N := \mathrm{span}\{\phi_i\}_{i=1}^N$.
- $P_N : \mathcal{H} \to H_N$ a bounded projection map.
- Define $\mathcal{A}_N := P_N \mathcal{A} P_N$.
- Seek $u_N \in H_N, \lambda_N \in \mathbb{C}$ s.t.

$$\mathcal{A}_N u_N = \lambda_N u_N$$

- We represent $u_N = \sum_{i=1}^N \mathrm{u}_N^{(i)} \phi_i$.

Approximation space $H_N = \mathrm{span}\{\phi_i\}_{i=1}^N$

Methods depend on subspace choices

- Finite difference methods: ϕ_i are Lagrange interpolants.
- Spectral methods: ϕ_i are eigenfunctions of some operator, $\|u - P_N u\|_X \to 0$ very fast
- Method of particular solutions: ϕ_i are eigensolutions of operator in simpler geometries
- Method of moments: $\phi_i := \mathcal{A}^{i-1} u$ for some $u \in \mathcal{H}$
- Finite volume methods
- Finite element methods: ϕ are piecewise polynomials

Choices of H_N

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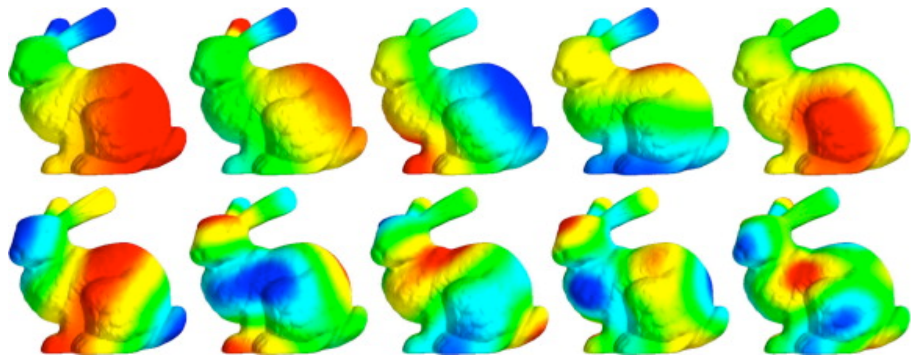

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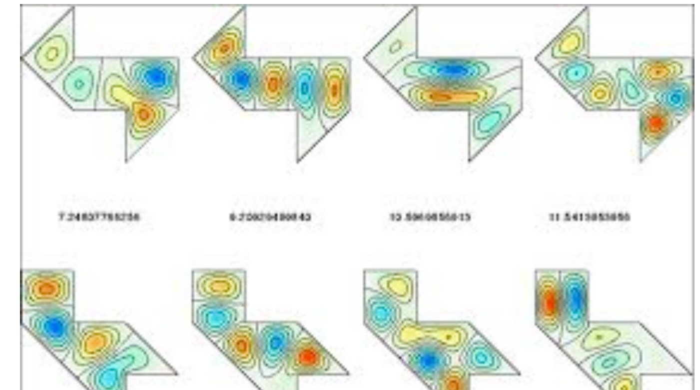

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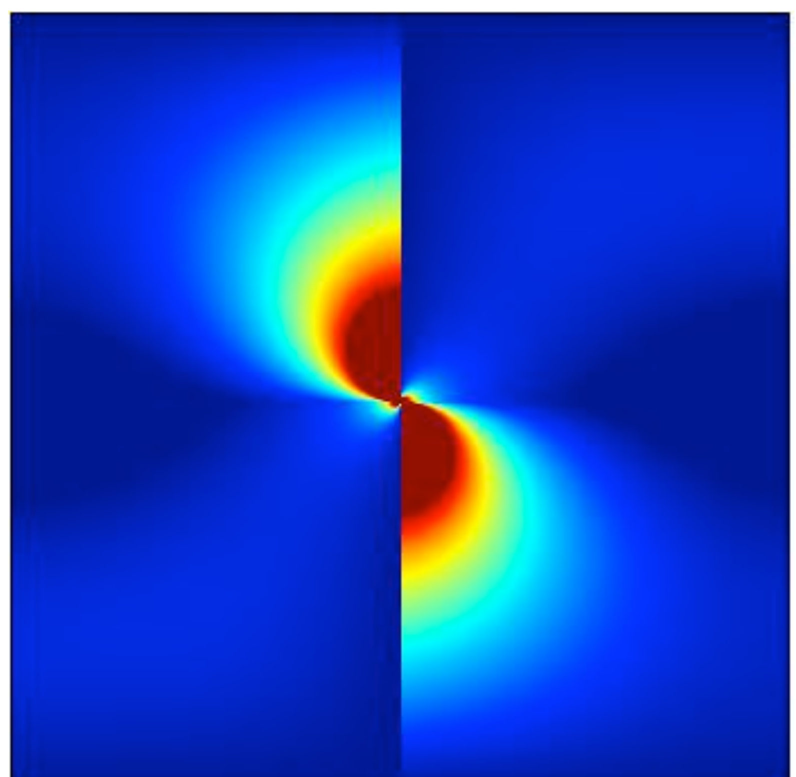

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Choices of H_N

Must ensure consistency, stability and convergence of approximations at this level

Recasting as a problem in Euclidean space

Infinite dim. function space	$\mathcal{A} : \mathrm{dom}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$	Solve $\mathcal{A}u = \lambda u$
Finite dim. function space	$\mathcal{A}_N : H_N \to H_N$	Solve $\mathcal{A}_N u_N = \lambda u_N$
Finite dim. Eucl. space	$\mathtt{A}_{NM}, \mathtt{B}_{NM} : \mathbb{C}^N \to \mathbb{C}^M$	Solve $\mathtt{A}_{NM}\mathtt{u}_N = \mathtt{B}_{NM}(\Lambda_N)\mathtt{u}_N$

Third choice: how to satisfy equation

Have $H_N = \mathrm{span}\{\phi_i\}_N, u_N = \sum_{i=1}^{N} \mathtt{u}_N^{(i)} \phi_i$. How to get

$$\mathtt{A}_{NM}\mathtt{u}_N = \mathtt{B}_{NM}(\Lambda_N)\mathtt{u}_N?$$

Method of weighted residuals. Let $\{\psi_j\}_{j=1}^{M}$ be linearly independent.

$$\mathrm{minimize}_{\mathtt{w}} \left\| \langle \mathcal{A}_N \sum_{i=1}^{N} \mathtt{w}_N^{(i)} \phi_i - \lambda_N \sum_{i=1}^{N} \mathtt{w}_N^{(i)} \phi_i, \psi_j \rangle \right\|_{\mathcal{W}}$$

Size and elements of $\mathtt{A}_{NM}, \mathtt{B}_{NM}$ depend on this choice.

Third choice: how to satisfy equation

Have $H_N = \mathrm{span}\{\phi_i\}_N$, $u_N = \sum_{i=1}^{N} \mathtt{u}_N^{(i)} \phi_i$. How to get

$$\mathtt{A}_{NM}\mathtt{u}_N = \mathtt{B}_{NM}(\Lambda_N)\mathtt{u}_N?$$

Third choice: how to satisfy equation

Have $H_N = \mathrm{span}\{\phi_i\}_N$, $u_N = \sum_{i=1}^{N} \mathtt{u}_N^{(i)} \phi_i$. How to get

$$\mathtt{A}_{NM}\mathtt{u}_N = \mathtt{B}_{NM}(\Lambda_N)\mathtt{u}_N?$$

- Collocation methods: $\mathcal{H}$ a Banach space, enforce $\langle \mathcal{A}_N u_N, \delta(x_j)\rangle = \lambda_N \langle u_N, \delta(x_j)\rangle$.
- Galerkin methods: $\mathcal{H}$ a Banach space. Use ψ_j in H_N^*. Enforce $\langle \mathcal{A}_N u_N, \psi_j\rangle_H = \lambda_N \langle u_N, \psi_j\rangle_H$.
- (Orthogonal) Galerkin methods: $\mathcal{H}$ a Hilbert space. Use $\psi_i in H_N$. Enforce $(\mathcal{A}_N u_N, \psi_j)_H = \lambda_N (u_N, \psi_j)_H$.

Size and elements of $\mathtt{A}_{NM}, \mathtt{B}_{NM}$ depend on this choice.

Third choice: how to satisfy equation

Must ensure consistency, stability and convergence at this level as well.

Numerics

from: Boffi, Gardini, Gastaldi, 2012.

Numerics

Some Remarks on Eigenvalue Approximation by Finite Elements 5

Table 1 Eigenvalues computed using the code in Listing 1.1 for different values of n

	Computed (rate)				
	$n = 8$	$n = 16$	$n = 32$	$n = 64$	$n = 128$
	1.0129	1.0032 (2.0)	1.0008 (2.0)	1.0002 (2.0)	1.0001 (2.0)
	4.2095	4.0517 (2.0)	4.0129 (2.0)	4.0032 (2.0)	4.0008 (2.0)
	10.0803	9.2631 (2.0)	9.0652 (2.0)	9.0163 (2.0)	9.0041 (2.0)
	19.4537	16.8382 (2.0)	16.2067 (2.0)	16.0515 (2.0)	16.0129 (2.0)
	33.2628	27.0649 (2.0)	25.5059 (2.0)	25.1257 (2.0)	25.0314 (2.0)
	51.3724	40.3212 (1.8)	37.0525 (2.0)	36.2610 (2.0)	36.0651 (2.0)
	69.5582	57.0672 (1.3)	50.9572 (2.0)	49.4840 (2.0)	49.1206 (2.0)
		77.8147	67.3528 (2.0)	64.8266 (2.0)	64.2059 (2.0)
		103.0473	86.3943 (2.0)	82.3258 (2.0)	81.3299 (2.0)
		133.0513	108.2597 (2.0)	102.0237 (2.0)	100.5030 (2.0)
DOF	7	15	31	63	127

from: Boffi, Gardini, Gastaldi, 2012.

Numerics

Table 2 Eigenvalues computed using the code in Listing 1.2 for different values of n

	Computed (rate)				
	$n = 8$	$n = 16$	$n = 32$	$n = 64$	$n = 128$
	1.0000	1.0000 (4.0)	1.0000 (4.0)	1.0000 (4.0)	1.0000 (4.0)
	4.0020	4.0001 (4.0)	4.0000 (4.0)	4.0000 (4.0)	4.0000 (4.0)
	9.0225	9.0015 (3.9)	9.0001 (4.0)	9.0000 (4.0)	9.0000 (4.0)
	16.1204	16.0082 (3.9)	16.0005 (4.0)	16.0000 (4.0)	16.0000 (4.0)
	25.4327	25.0307 (3.8)	25.0020 (3.9)	25.0001 (4.0)	25.0000 (4.0)
	37.1989	36.0899 (3.7)	36.0059 (3.9)	36.0004 (4.0)	36.0000 (4.0)
	51.6607	49.2217 (3.6)	49.0148 (3.9)	49.0009 (4.0)	49.0001 (4.0)
	64.8456	64.4814 (0.8)	64.0328 (3.9)	64.0021 (4.0)	64.0001 (4.0)
	95.7798	81.9488 (4.0)	81.0659 (3.8)	81.0042 (4.0)	81.0003 (4.0)
	124.9301	101.7308 (3.8)	100.1229 (3.8)	100.0080 (3.9)	100.0005 (4.0)
DOF	15	31	63	127	255

from: Boffi, Gardini, Gastaldi, 2012.

Numerics

Table 5 Eigenvalues computed using the code in Listing 1.5 for different values of n

Exact	Computed (rate)				
	$n = 8$	$n = 16$	$n = 32$	$n = 64$	$n = 128$
	1.0129	1.0032 (2.0)	1.0008 (2.0)	1.0002 (2.0)	1.0001 (2.0)
	4.2095	4.0517 (2.0)	4.0129 (2.0)	4.0032 (2.0)	4.0008 (2.0)
	10.0803	9.2631 (2.0)	9.0652 (2.0)	9.0163 (2.0)	9.0041 (2.0)
	19.4537	16.8382 (2.0)	16.2067 (2.0)	16.0515 (2.0)	16.0129 (2.0)
	33.2628	27.0649 (2.0)	25.5059 (2.0)	25.1257 (2.0)	25.0314 (2.0)
	51.3724	40.3212 (1.8)	37.0525 (2.0)	36.2610 (2.0)	36.0651 (2.0)
	69.5582	57.0672 (1.3)	50.9572 (2.0)	49.4840 (2.0)	49.1206 (2.0)
	77.8147	77.8147 (0.0)	67.3528 (2.0)	64.8266 (2.0)	64.2059 (2.0)
		103.0473	86.3943 (2.0)	82.3258 (2.0)	81.3299 (2.0)
		133.0513	108.2597 (2.0)	102.0237 (2.0)	100.5030 (2.0)
DOF	8	16	32	64	128

from: Boffi, Gardini, Gastaldi, 2012.

Numerics

All are FEM methods for Dirichlet problem on $[0,\pi]$

from: Boffi, Gardini, Gastaldi, 2012.

What can you trust?

Table 4 Eigenvalues computed using the code in Listing 1.4 for different values of n

Exact	Computed (rate)				
	$n = 8$	$n = 16$	$n = 32$	$n = 64$	$n = 128$
	0.0000	−0.0000	0.0000	0.0000	−0.0000
	1.0001	1.0000 (4.1)	1.0000 (4.0)	1.0000 (4.0)	1.0000 (4.0)
	3.9660	3.9981 (4.2)	3.9999 (4.0)	4.0000 (4.0)	4.0000 (4.0)
	7.4257	8.5541	8.8854	8.9711	8.9928
	8.7603	8.9873 (4.2)	8.9992 (4.1)	9.0000 (4.0)	9.0000 (4.0)
	14.8408	15.9501 (4.5)	15.9971 (4.1)	15.9998 (4.0)	16.0000 (4.0)
	16.7900	24.5524 (4.2)	24.9780 (4.3)	24.9987 (4.1)	24.9999 (4.0)
	38.7154	29.7390	34.2165	35.5415	35.8846
	39.0906	35.0393 (1.7)	35.9492 (4.2)	35.9970 (4.1)	35.9998 (4.0)
		46.7793	48.8925 (4.4)	48.9937 (4.1)	48.9996 (4.0)

What can you trust?

Table 4 Eigenvalues computed using the code in Listing 1.4 for different values of n

Exact	Computed (rate)				
	$n = 8$	$n = 16$	$n = 32$	$n = 64$	$n = 128$
	0.0000	−0.0000	0.0000	0.0000	−0.0000
	1.0001	1.0000 (4.1)	1.0000 (4.0)	1.0000 (4.0)	1.0000 (4.0)
	3.9660	3.9981 (4.2)	3.9999 (4.0)	4.0000 (4.0)	4.0000 (4.0)
	7.4257	8.5541	8.8854	8.9711	8.9928
	8.7603	8.9873 (4.2)	8.9992 (4.1)	9.0000 (4.0)	9.0000 (4.0)
	14.8408	15.9501 (4.5)	15.9971 (4.1)	15.9998 (4.0)	16.0000 (4.0)
	16.7900	24.5524 (4.2)	24.9780 (4.3)	24.9987 (4.1)	24.9999 (4.0)
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		46.7793	48.8925 (4.4)	48.9937 (4.1)	48.9996 (4.0)

What can you trust?

Table 6 Eigenvalues computed using the code in Listing 1.6 for different values of n

Exact	Computed (rate with respect to 6λ)				
	$n = 8$	$n = 16$	$n = 32$	$n = 64$	$n = 128$
	5.7061	5.9238 (1.9)	5.9808 (2.0)	5.9952 (2.0)	5.9988 (2.0)
	19.8800	22.8245 (1.8)	23.6953 (1.9)	23.9231 (2.0)	23.9807 (2.0)
	36.7065	48.3798 (1.6)	52.4809 (1.9)	53.6123 (2.0)	53.9026 (2.0)
	51.8764	79.5201 (1.4)	91.2978 (1.8)	94.7814 (1.9)	95.6925 (2.0)
	63.6140	113.1819 (1.2)	138.8165 (1.7)	147.0451 (1.9)	149.2506 (2.0)
	71.6666	146.8261 (1.1)	193.5192 (1.6)	209.9235 (1.9)	214.4494 (2.0)
	76.3051	178.6404 (0.9)	253.8044 (1.5)	282.8515 (1.9)	291.1344 (2.0)
	77.8147	207.5058 (0.8)	318.0804 (1.4)	365.1912 (1.8)	379.1255 (1.9)
		232.8461	384.8425 (1.3)	456.2445 (1.8)	478.2172 (1.9)
		254.4561	452.7277 (1.2)	555.2659 (1.7)	588.1806 (1.9)
DOF	8	16	32	64	128

$\lambda = 19.88003$

What can you trust?

Table 6 Eigenvalues computed using the code in Listing 1.6 for different values of n

Exact	Computed (rate with respect to 6λ)				
	$n = 8$	$n = 16$	$n = 32$	$n = 64$	$n = 128$
1	5.7061	5.9238 (1.9)	5.9808 (2.0)	5.9952 (2.0)	5.9988 (2.0)
4	19.8800	22.8245 (1.8)	23.6953 (1.9)	23.9231 (2.0)	23.9807 (2.0)
9	36.7065	48.3798 (1.6)	52.4809 (1.9)	53.6123 (2.0)	53.9026 (2.0)
16	51.8764	79.5201 (1.4)	91.2978 (1.8)	94.7814 (1.9)	95.6925 (2.0)
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64	77.8147	207.5058 (0.8)	318.0804 (1.4)	365.1912 (1.8)	379.1255 (1.9)
81		232.8461	384.8425 (1.3)	456.2445 (1.8)	478.2172 (1.9)
100		254.4561	452.7277 (1.2)	555.2659 (1.7)	588.1806 (1.9)
DOF	8	16	32	64	128

$\lambda=19.88003$

Numerics can be dangerous!

Any method could exhibit these problems [Fox, Henrici, Moler, 1967], method of particular solutions.
Fixed: [Betcke and Trefethen, 2005, Barnett...]

474 TIMO BETCKE AND LLOYD N. TREFETHEN

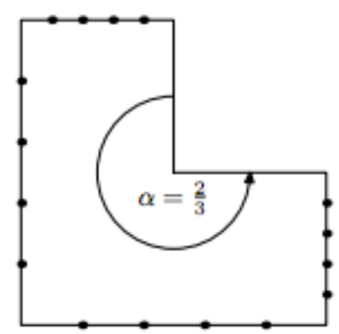

Fig. 3.2 *The L-shaped membrane, with N collocation points equally spaced along each side not adjacent to the reentrant corner (here $N = 4$).*

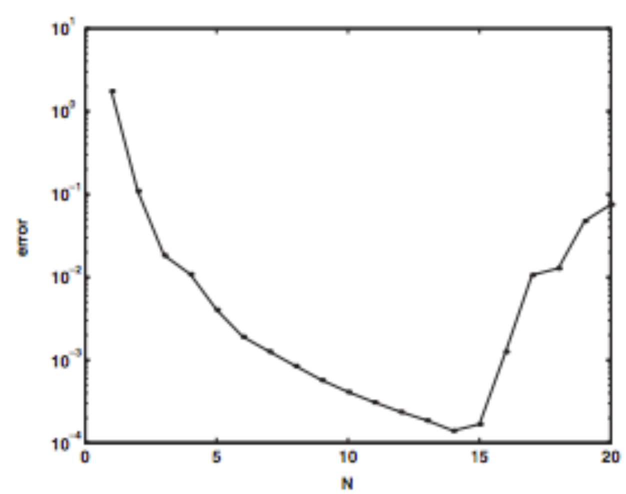

Fig. 3.3 *The error of FHM for the first eigenvalue of the L-shaped domain. Convergence breaks down after $N = 14$, and one never gets more than four digits of accuracy.*

MPS error decreases, then increases in N.

Discrete eigenvalue algorithms

Let's examine the consequences of some choices.

- Formulate problem in Hilbert space, $\mathcal{H}$, operator $\mathcal{A}$
- Let $H_N = \mathrm{span}\{\phi_n\}_{n=1}^N$, orthonormal basis
- Use projection P_N, $\mathcal{A}_N := P_N \mathcal{A} P_N$
- $\mathcal{A}_N u_N = \lambda_N u_N, \quad u_N = \sum_{i=1}^N \mathtt{u}_N^{(i)} \phi_i$. Write $\mathtt{u}_N = (\mathtt{u}_N^{(1)}, \mathtt{u}_N^{(2)}, \ldots, \mathtt{u}_N^{(1)})^T$.
- Use Galerkin method $(A_{NN})_{i,j} := (\mathcal{A}_N \phi_i, \phi_j)_H$
- Spectral problem in Euclidean space: $\mathtt{A}_{NN} \mathtt{u}_N = \Lambda_N \mathtt{u}_N$

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- Use Galerkin method $(A_{NN})_{i,j} := (\mathcal{A}_N \phi_i, \phi_j)_H$
- Spectral problem in Euclidean space: $\mathtt{A}_{NN} \mathtt{u}_N = \Lambda_N \mathtt{u}_N$

If $N > 5$, need iterative methods to approximate Λ_N.

Classical iterations for $\mathtt{A}_{NN}\mathtt{u}_N = \Lambda_N \mathtt{u}_N$.

- Power iteration: Start with arbitrary $\mathtt{b}$. Columns of Krylov matrix

$$\mathcal{K}_n = [\mathtt{b}, \mathtt{A}_{NN}\mathtt{b}\mathtt{A}^2_{NN}\mathtt{b}....\mathtt{A}_{NN}\mathtt{b}$$

  converge to the e.v. for the largest e.value.
- Arnoldi iteration: Use stabilized Gram-Schmidt to generate orthogonal vectors $\mathtt{q}_{N,i}$,

$$span\{\mathtt{q}_{N,1}, \mathtt{q}_{N,2}, ...\mathtt{q}_{N,n}\} = span\{\mathcal{K}_n\}$$

- Lanczos iteration: Arnoldi iteration for symmetric matrices. Converts $\mathtt{A}_{NN} \to T_{nn}$, tridiagonal.

[Golub and van Loan, 1996, Trefethen and Bau, 1997...]

Typical theorems

- If $\theta_i^n, \Lambda_{N,i}$ are the *ith* eigenvalues of T_{nn} and $\mathtt{A}_{NN}$, then get bounds on $|\theta_i^n - \Lambda_{N,i}$ in terms of spectral gaps.
  [Kaniel-Paige-Saad bounds]

Typical theorems

- If $\theta_i^n, \Lambda_{N,i}$ are the *ith* eigenvalues of T_{nn} and $\mathtt{A}_{NN}$, then get bounds on $|\theta_i^n - \Lambda_{N,i}$ in terms of spectral gaps. [Kaniel-Paige-Saad bounds]
- Suppose we compute the eigenpair $(\tilde{\mathtt{u}}_N, \tilde{\Lambda}_N)$. Then,

$$\mathtt{A}_{NN}\tilde{\mathtt{u}}_N = \tilde{\Lambda}_N\tilde{\mathtt{u}}_N + r.$$

Backward stability means $\exists$ real symmetric matrices $\mathtt{E}$ such that

$$(\mathtt{A}_{NN} + \mathtt{E})\tilde{\mathtt{u}}_N = \tilde{\Lambda}_N\tilde{\mathtt{u}}_N.$$

If r is small in norm, then it can be shown that

$$|\tilde{\Lambda}_N - \Lambda_{N,i}| \leq \frac{\|r\|_2^2}{\delta}, \qquad \delta := \min_{k \neq i} |\tilde{\Lambda} - \Lambda_{N,i}|.$$

The solid angle θ between the computed eigenvector $\tilde{\mathtt{u}}_N$ and the actual eigenvector $\mathtt{u}_N$ satisfies

$$\sin(\theta) \leq \frac{\|r\|_2}{\delta}.$$

Errors to examine: eigenvector

$$\|u - \sum_{i=1}^{N} \tilde{\mathtt{w}}_N^{(i)} \phi_i\| \leq \underbrace{\|u - P_N u\|}_{\text{A}} + \underbrace{\|P_N u - u_N\|}_{\text{B}} + \underbrace{\|u_N - \sum_{i=1}^{N} \tilde{\mathtt{u}}_N^{(i)} \phi_i\|}_{\text{C}}$$

$$+ \underbrace{\|\sum_{i=1}^{N} [\tilde{\mathtt{w}}_N^{(i)} - \tilde{\mathtt{u}}_N^{(i)}] \phi_i\|}_{\text{D}}$$

$$\leq \alpha(N)\|u\| \to 0, \quad \beta(N)\|\sum_{i=1}^{N} \tilde{\mathtt{w}}_N^{(i)} \phi_i\| \to 0$$

- A: Best approximation error (how 'good' is H_N, P_N?)
- B: Error in H_N
- C: Approximation using numerical linear algebra
- D: Rounding arithmetic.

(A very brief) Literature

- I. Babuska and J. Osborn (1991), Eigenvalue problems. In Handbook of Numerical Analysis, Vol. II, North-Holland, Amsterdam, pp. 641787.
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(A very brief) Literature

- V. Bonnaillie-Noel, M. Dauge, N. Popoff, N. Raymond Discrete spectrum of a model Schrödinger operator on the half-plane with Neumann conditions,Zeitschrift fr angewandte Mathematik und Physik (ZAMP) Volume 63, Issue 2 (2012), Page 203-231
- C. Moler and L. Payne, Bounds for eigenvalues and eigenvectors of symmetric operators, SIAM J. Numer. Anal., 5 (1968), pp. 6470.
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Spectral Problem 1

Let Ω be an open set in $\mathbb{R}^d$ with Lipschitz boundary.

Spectral problem 1, strong form

Find $(u, \lambda) \in (\mathcal{H}^2(\Omega), \mathbb{R})$ such that a.e.

$$-\Delta u = \lambda u \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega.$$

Spectral problem 1, formulation

Denote by $(\cdot,\cdot)$ the L^2 inner product on Ω

Spectral problem 1, minimization form

$$\lambda = \min_{w \in \mathcal{H}^1_0(\Omega), \|w\|_0 \neq 0} \frac{(\nabla w, \nabla w)}{(w, w)} = (\nabla u, \nabla u)$$

Spectral problem 1, formulation

Denote by $(\cdot, \cdot)$ the L^2 inner product on Ω

Spectral problem 1, minimization form

$$\lambda = \min_{w \in \mathcal{H}_0^1(\Omega), \|w\|_0 \neq 0} \frac{(\nabla w, \nabla w)}{(w, w)} = (\nabla u, \nabla u)$$

Equivalently:

Spectral problem 1, variational form

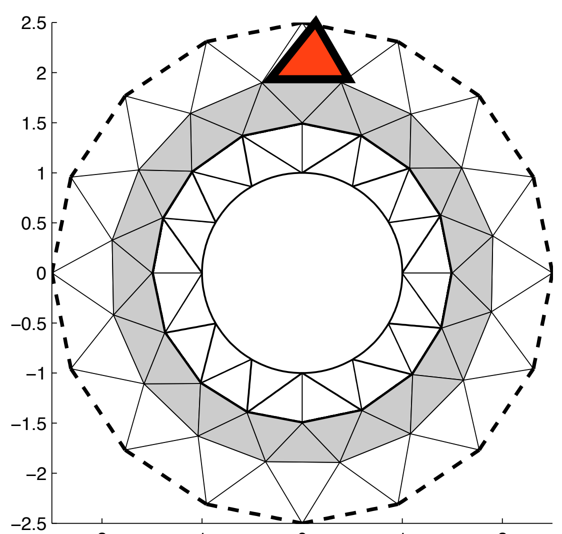
Find $(u, \lambda) \in (\mathcal{H}_0^1(\Omega), \mathbb{R})$ such that for all $v \in \mathcal{H}^1(\Omega)$,

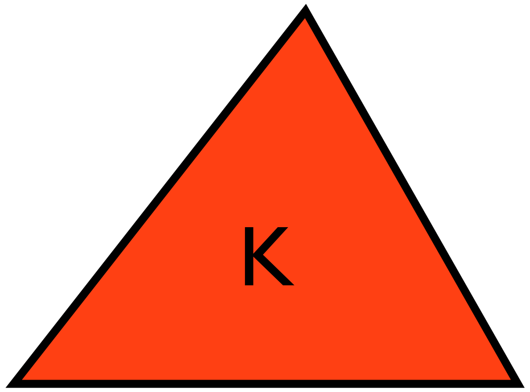
$$(\nabla u, \nabla v) = \lambda(u, v).$$

What is a finite element?

A finite element triple $(K, \mathcal{P}, \Sigma)$ consists of:

- A geometric domain K, used to tesselate/mesh a region in space;
- A finite dimensional vector space $\mathcal{P}$ on this domain, approximating some function space;
- A set of linear functionals Σ dual to the approximation space.



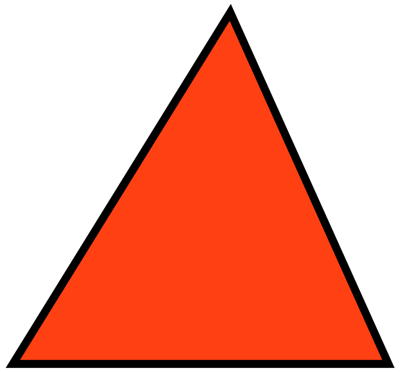


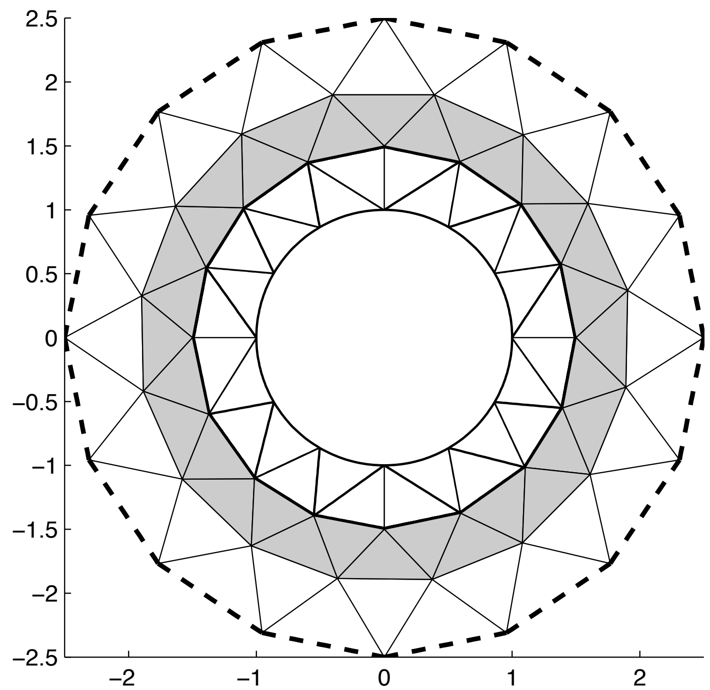
Conforming finite element

Let $\Pi_T = \{\tau_i\}_{i=1}^{T} = \overline{\Omega}$ be a triangulation.

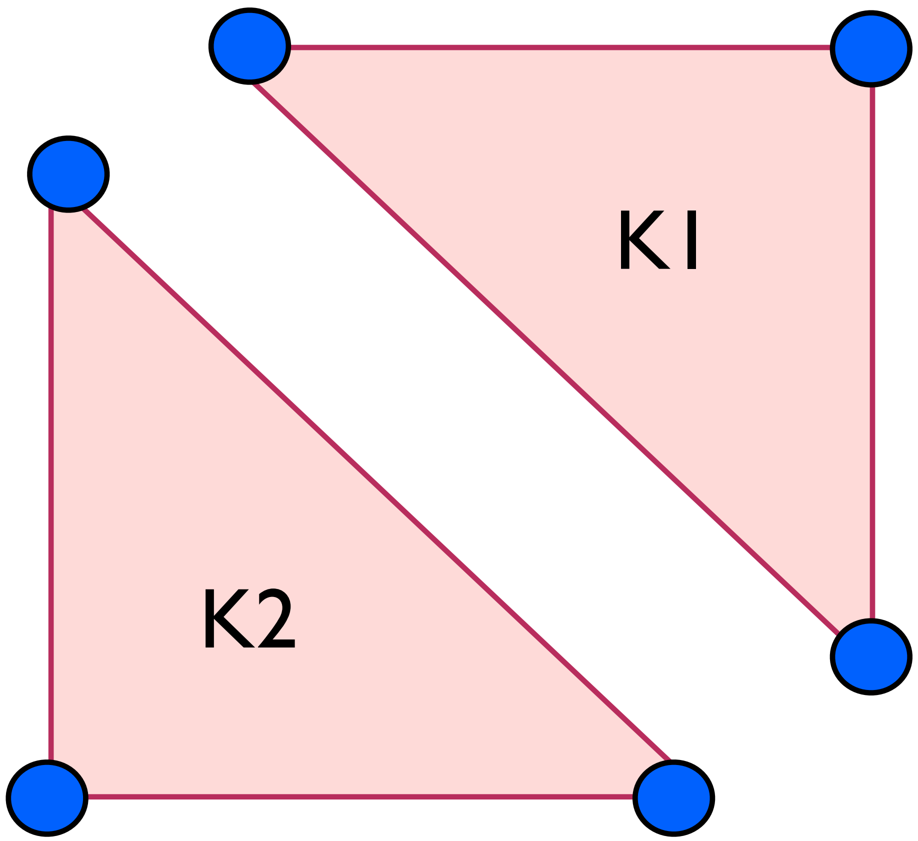
$$V_N := \left\{ w \in \mathcal{H}^1(\Omega) \middle|\ w|_{\tau_i} \in \mathcal{P}.\ \forall \tau_i \in \Pi_T,\ w \text{ is continuous} \right\}$$

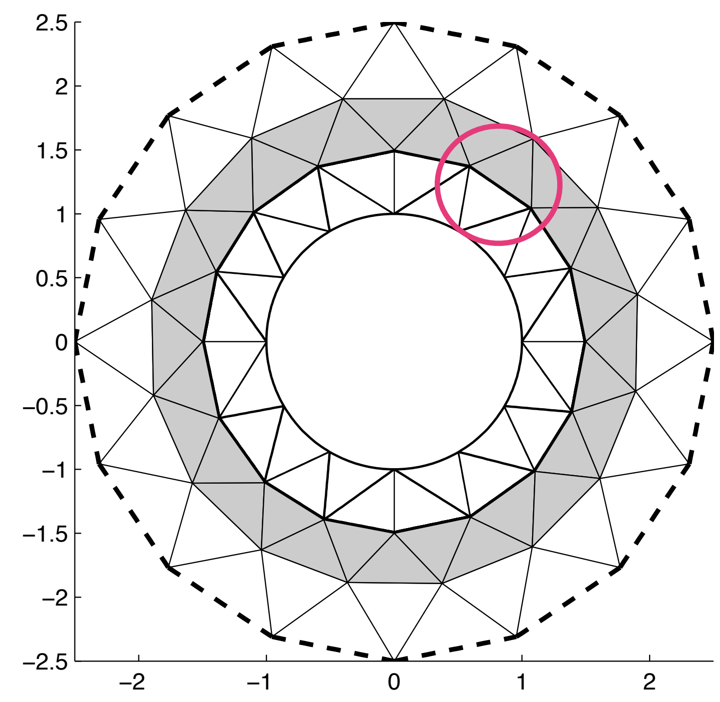
Clearly $V_T \subset \mathcal{H}^1(\Omega)$.





Conforming H1 linear elements





Conforming H1 linear elements: continuity across edges

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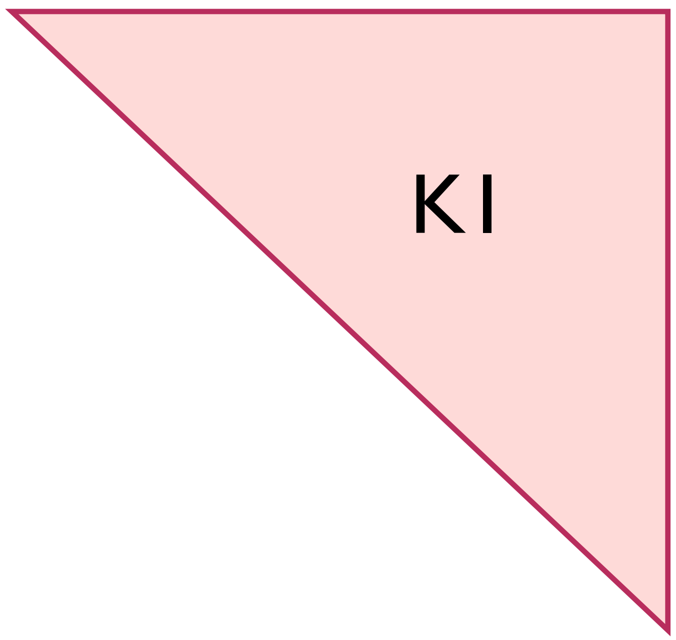
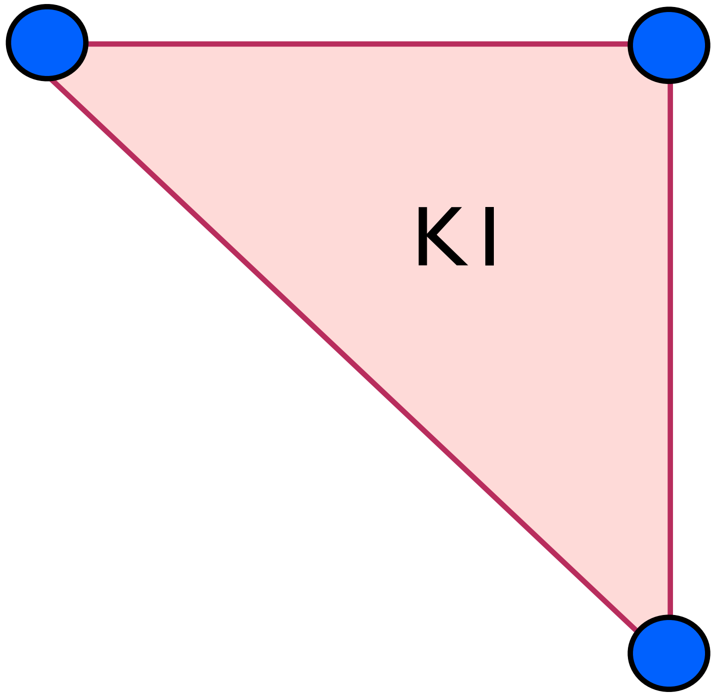


Conforming H1 linear elements: continuity across edges



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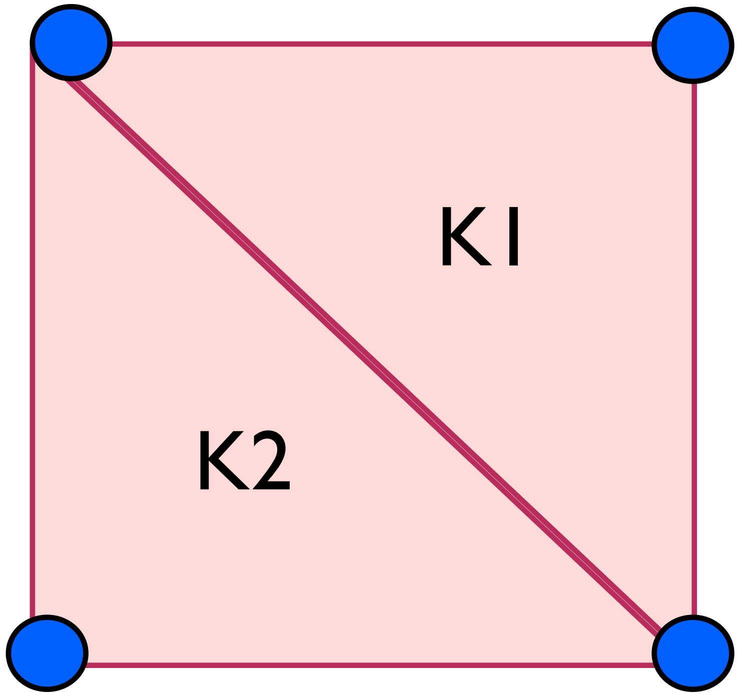

Conforming H1 linear elements and interpolation

Interpolate $f(x, y) = x^2 + y^2$

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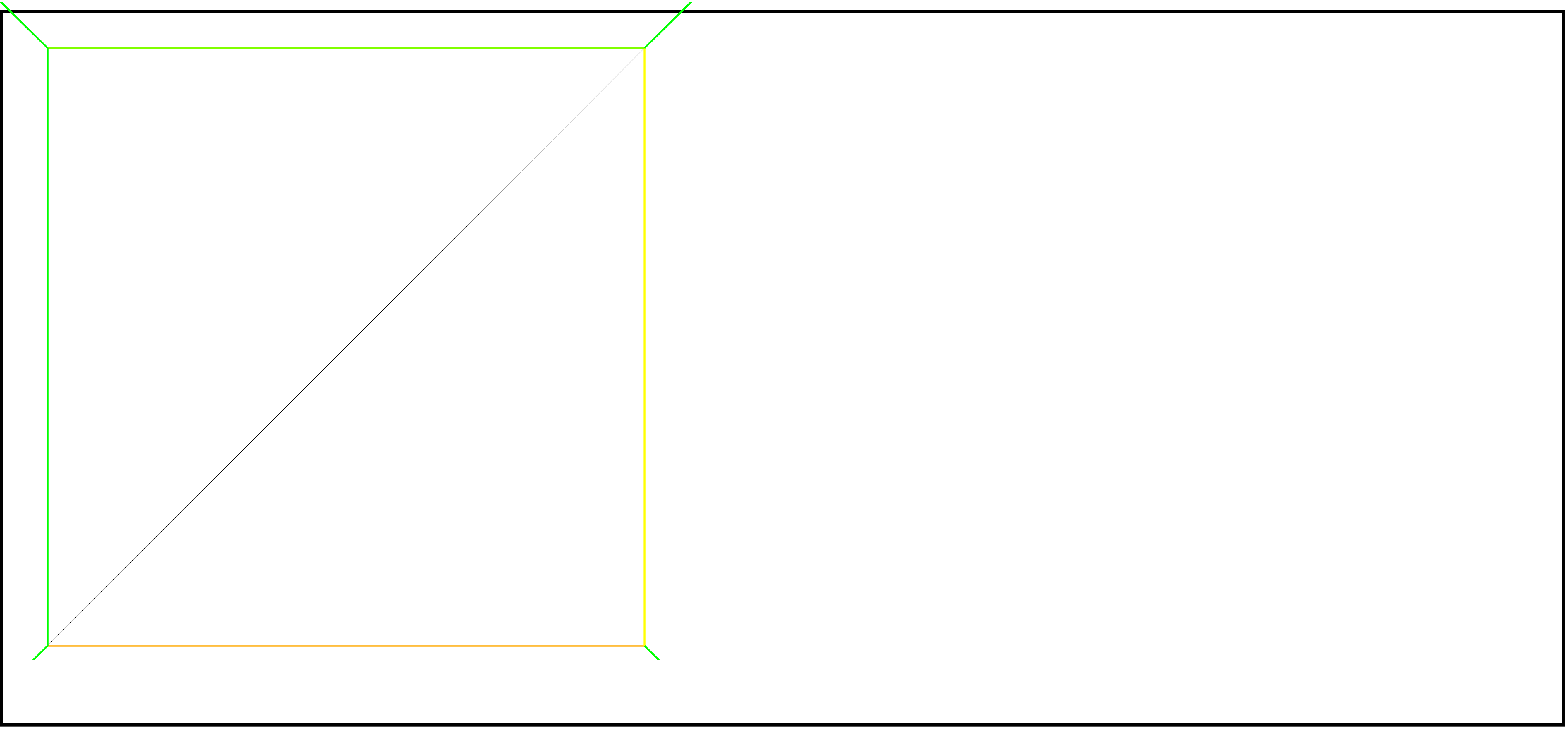

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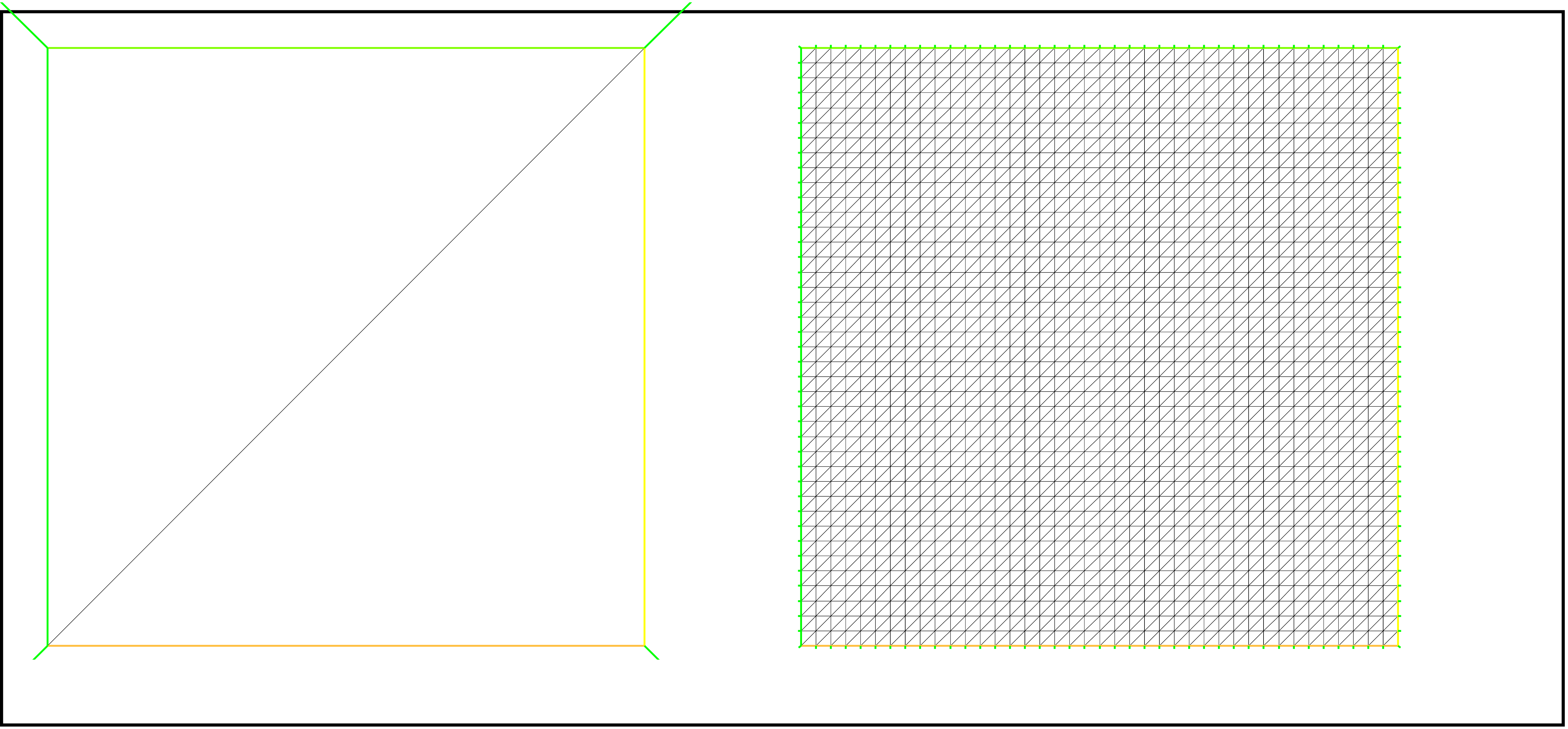

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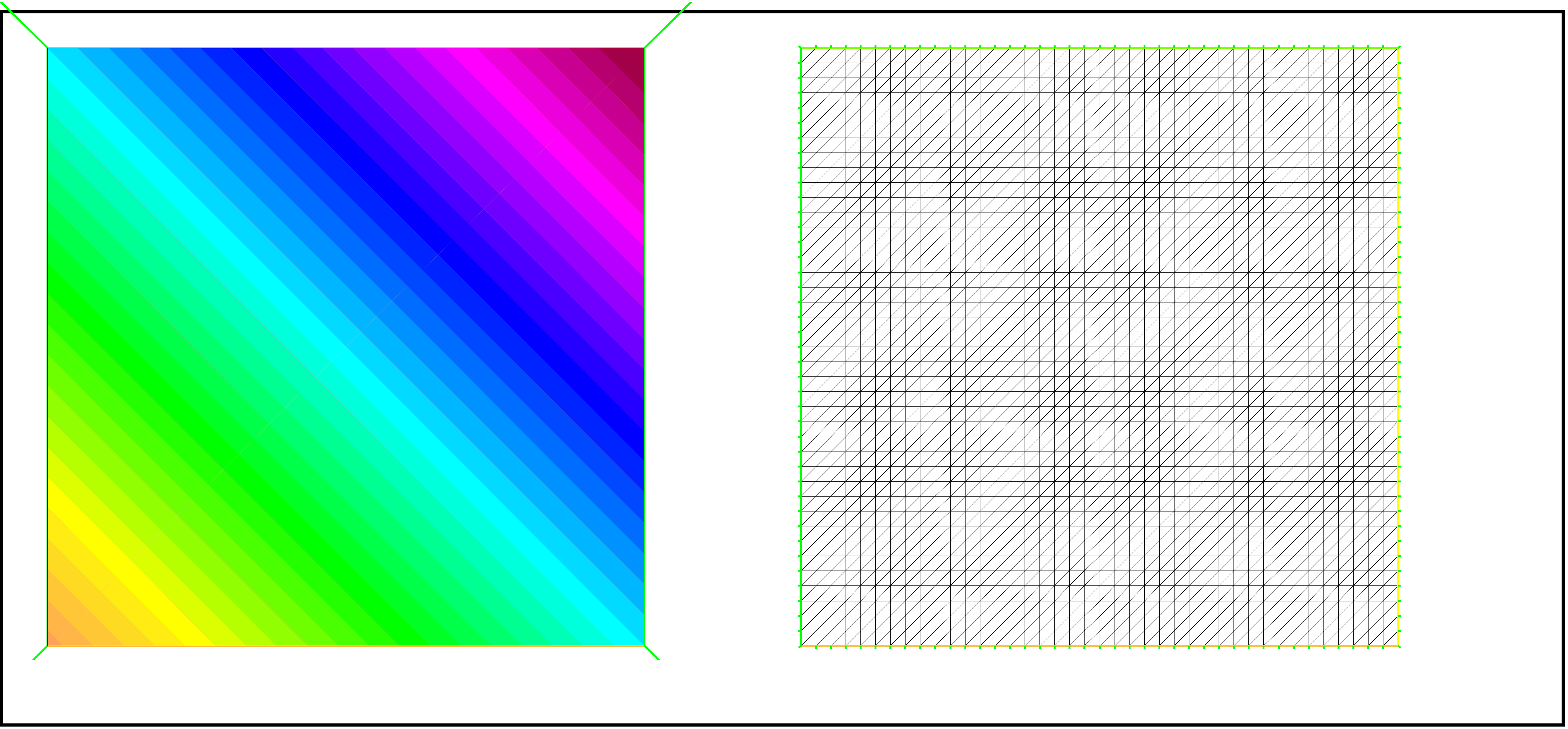

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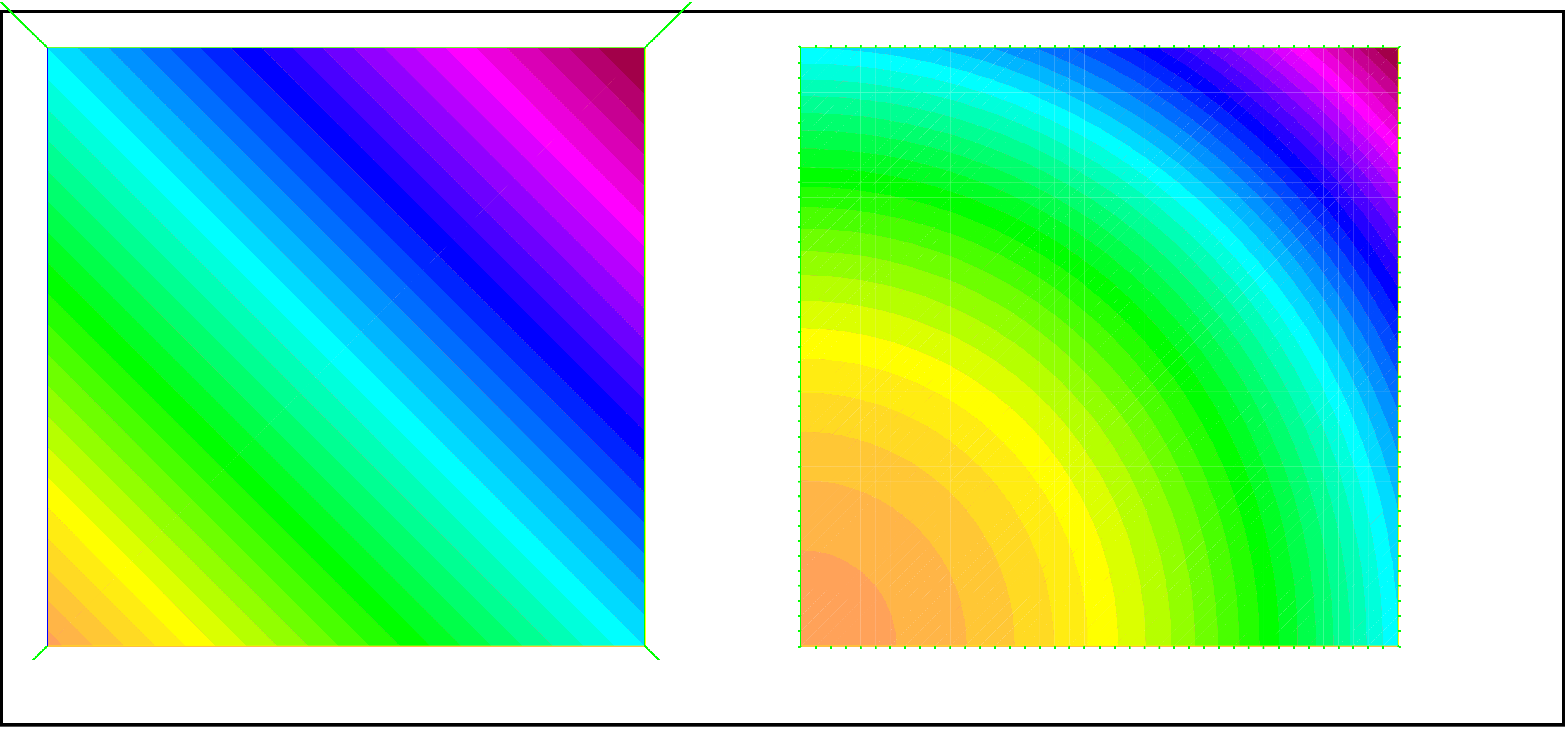

Best approximation by conforming FEM

Sample best approximation result

Under assumptions on Ω and the tessellation, the best approximation error of $u \in H^k(\Omega)$ by conforming FEM of piecewise polynomials of degree p and mesh parameter h

$$\inf_{w \in V_N} \|u - w\|_1 \leq C(p) h^{\mu - 1} \|u\|_{H^k(\Omega)}, \qquad \mu = \min(k, p+1).$$

[Widlund '76,Rannacher and Scott, '82]

Typical convergence results for eigenpairs

Convergence using conforming elements

Let $\epsilon_N(\lambda) := \sup_{u \in \text{ Scaled e.v. for } \lambda}\{\inf_{\chi \in V_N} \|u - \chi\|_1.\}$

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Let $\epsilon_N(\lambda) := \sup_{u \in \text{Scaled e.v. for } \lambda} \{\inf_{\chi \in V_N} \|u - \chi\|_1.\}$ If λ is a simple eigenvalue, $\exists$ constants $c_1, c_2, c_3 > 0$ such that

$$\|u - u_N\|_1 \leq c_1 \epsilon_N(\lambda), \qquad c_2 \epsilon_N^2 \leq \lambda_{N,c} - \lambda \leq c_3 \epsilon_N^2.$$

[Chatelin '75, '83, Babuska and Osborn '89]

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Here ϵ_N depends on the regularity of the true eigenfunctions, and characterizes an approximation error.

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The best approximation error of $u \in H^k(\Omega)$ by conforming FEM of piecewise polynomials of degree p and mesh parameter h

$$\inf_{w \in V_h} \|u - w\|_1 \leq C(p) h^{\mu - 1} \|u\|_{H^k(\Omega)}, \qquad \mu = \min(k, p+1).$$

[Widlund '76,Rannacher and Scott, '82]

Conforming FEM: approximation of λ from above

Consider

Model eigenvalue problem, discrete form

Find $(u_N, \lambda_{N,c}) \in (V_N, \mathbb{R})$ such that for all $v_N \in V_N$,

$$(\nabla u_N, \nabla v_N) = \lambda_{N,c}(u_N, v_N).$$

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$$(\nabla u_N, \nabla v_N) = \lambda_{N,c}(u_N, v_N).$$

Theorem: $\lambda_{N,c} \geq \lambda$.

Conforming FEM: approximation of λ from above

Consider

Model eigenvalue problem, discrete form

Find $(u_N, \lambda_{N,c}) \in (V_N, \mathbb{R})$ such that for all $v_N \in V_N$,

$$(\nabla u_N, \nabla v_N) = \lambda_{N,c}(u_N, v_N).$$

Theorem: $\lambda_{N,c} \geq \lambda$.

Proof:

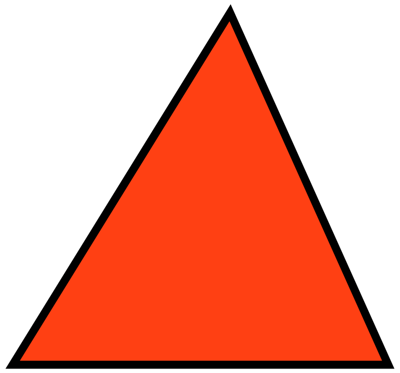
$$\lambda = \min_{w \in \mathcal{H}_0^1(\Omega), \|w\|_0 \neq 0} \frac{(\nabla w, \nabla w)}{(w, w)} \leq \min_{w_N \in V_N, \|w_N\|_0 \neq 0} \frac{(\nabla w_N, \nabla w_N)}{(w_N, w_N)} = \lambda_{N,c}.$$

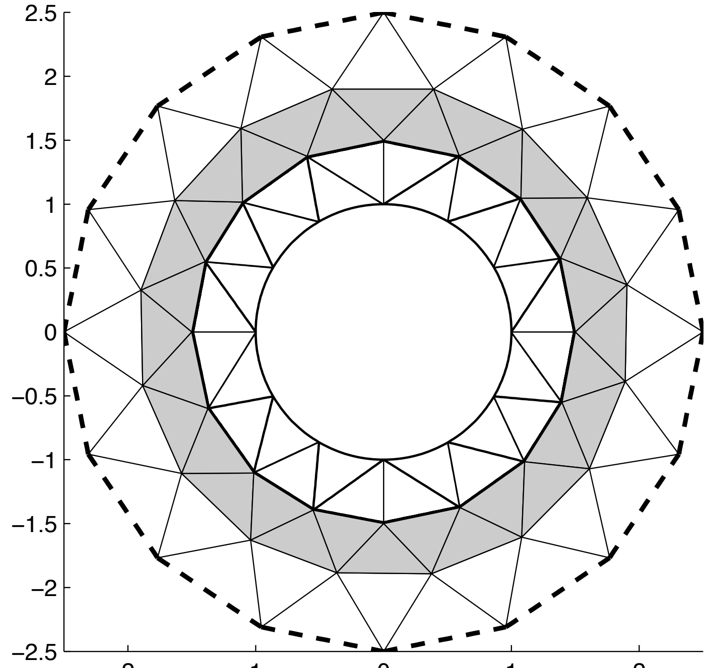
Non-conforming finite element

Let $\Pi_T = \{\tau_i\}_{i=1}^T = \overline{\Omega}$ be a triangulation.

$$V_N := \{w \in L^2(\Omega) \big| \; w|_{\tau_i} \in \mathcal{P} \; \forall \tau_i \in \Pi_T, \text{ } w \text{ is not continuous across edges, } w = 0 \text{ on nodes on boundary}\}$$

Clearly $V_N \not\subset \mathcal{H}^1(\Omega)$.





Non-conforming linear elements

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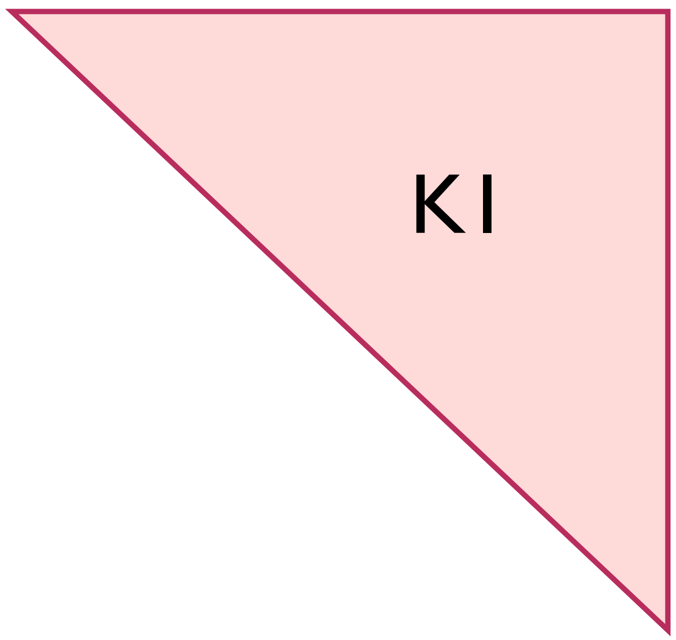
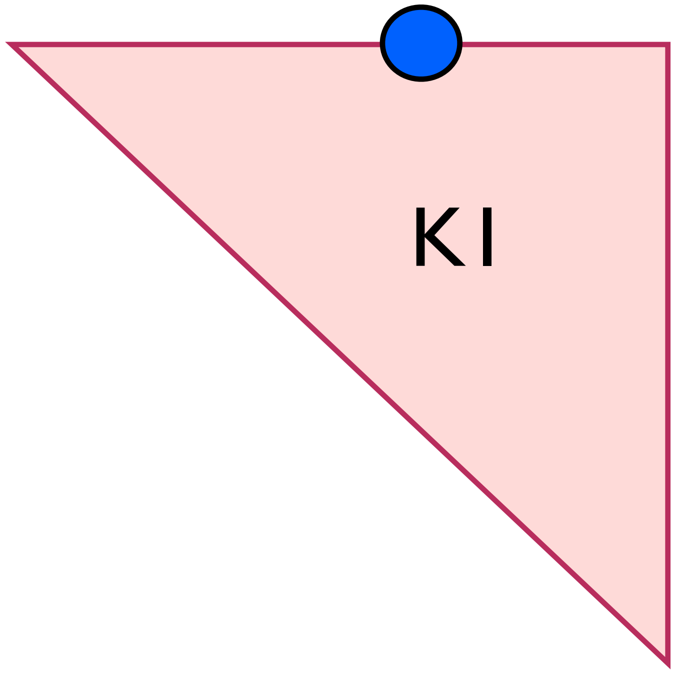


Non-conforming linear elements



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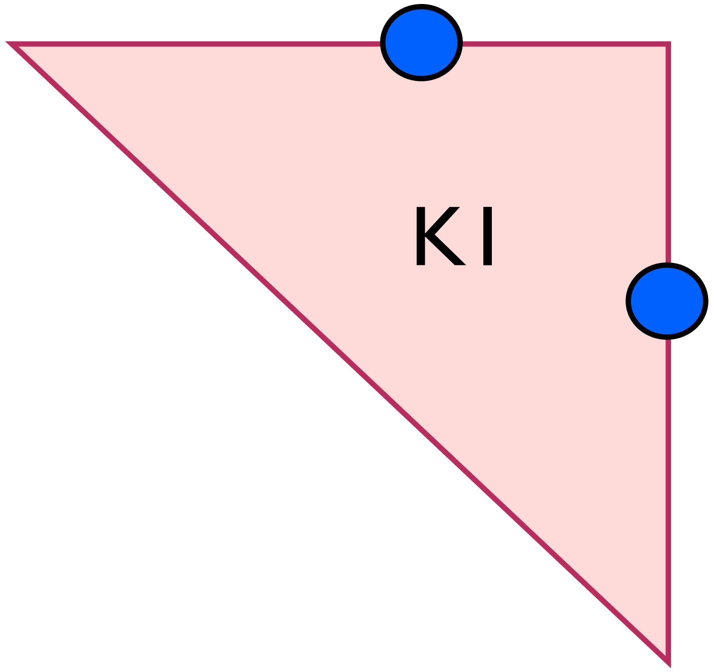

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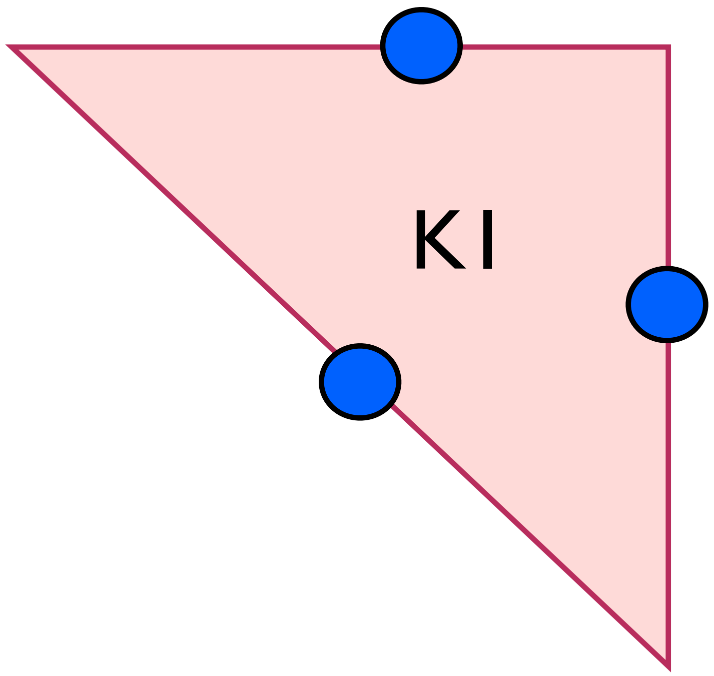

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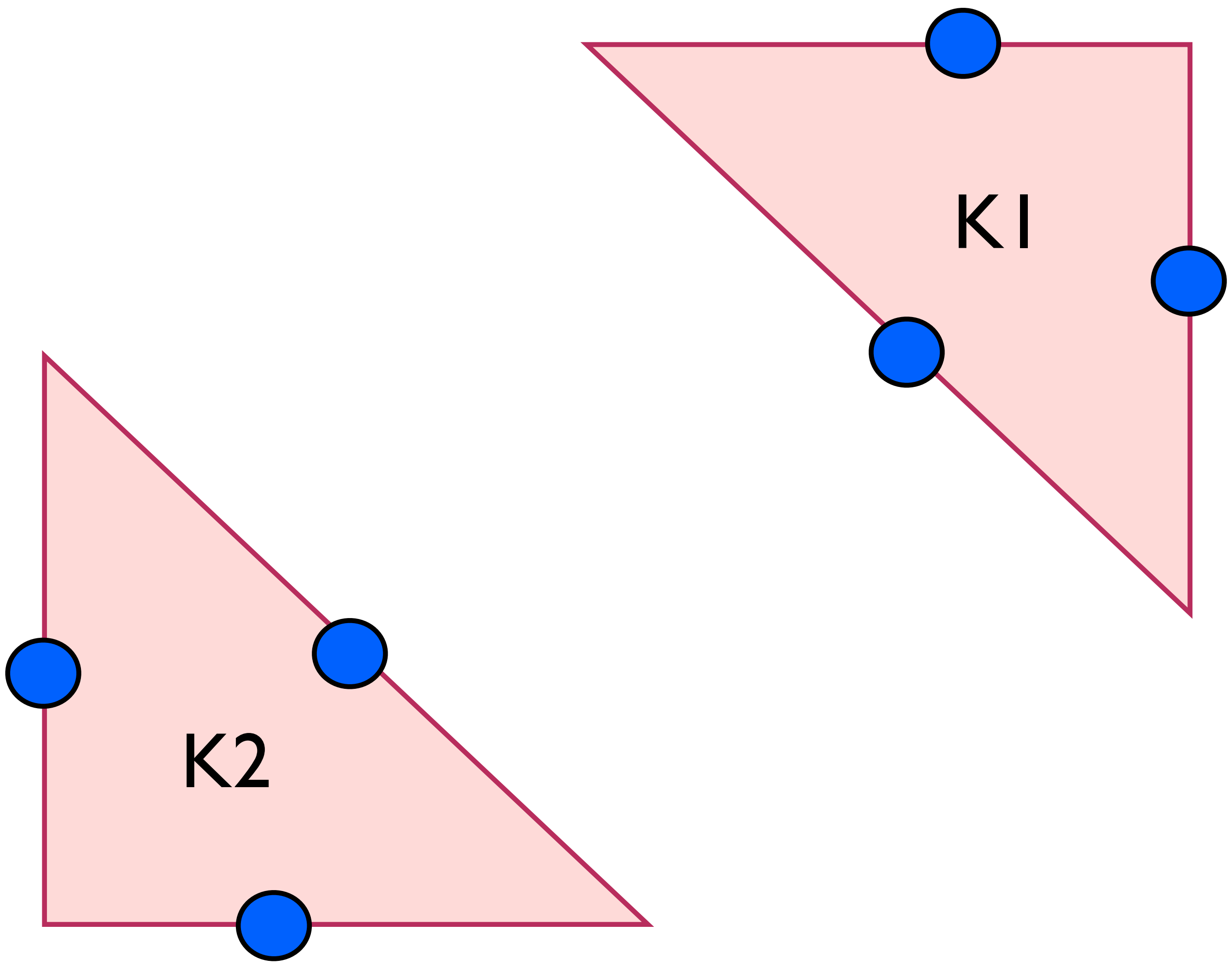

Non-conforming linear elements: continuity at midpoints

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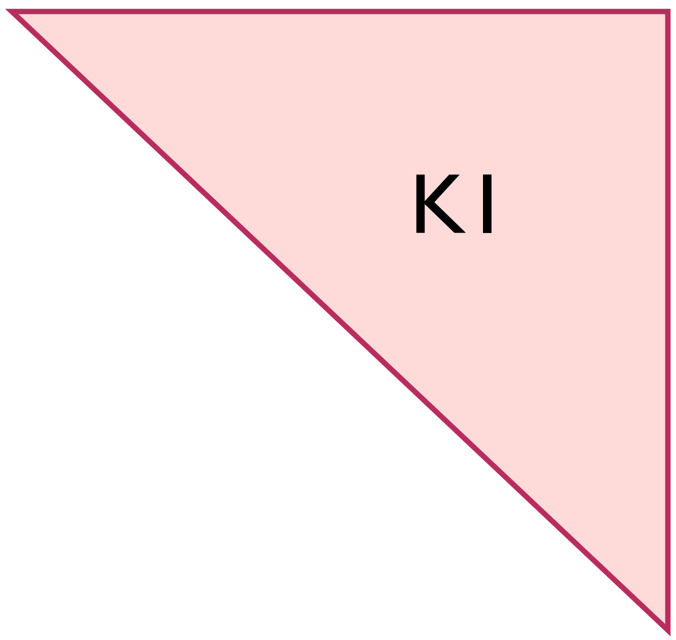

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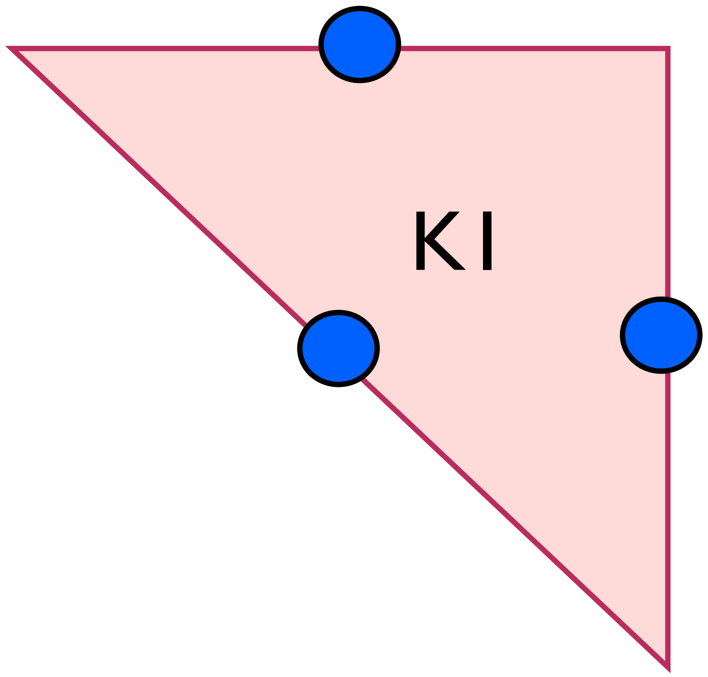
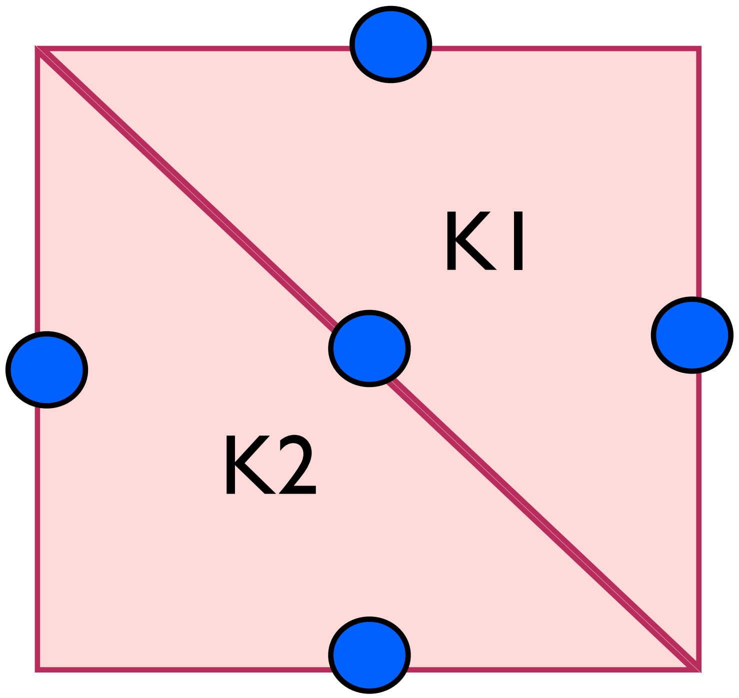


Non-conforming linear elements: continuity at midpoints



Non-conforming linear elements and interpolation

Interpolate $f(x, y) = x^2 + y^2$

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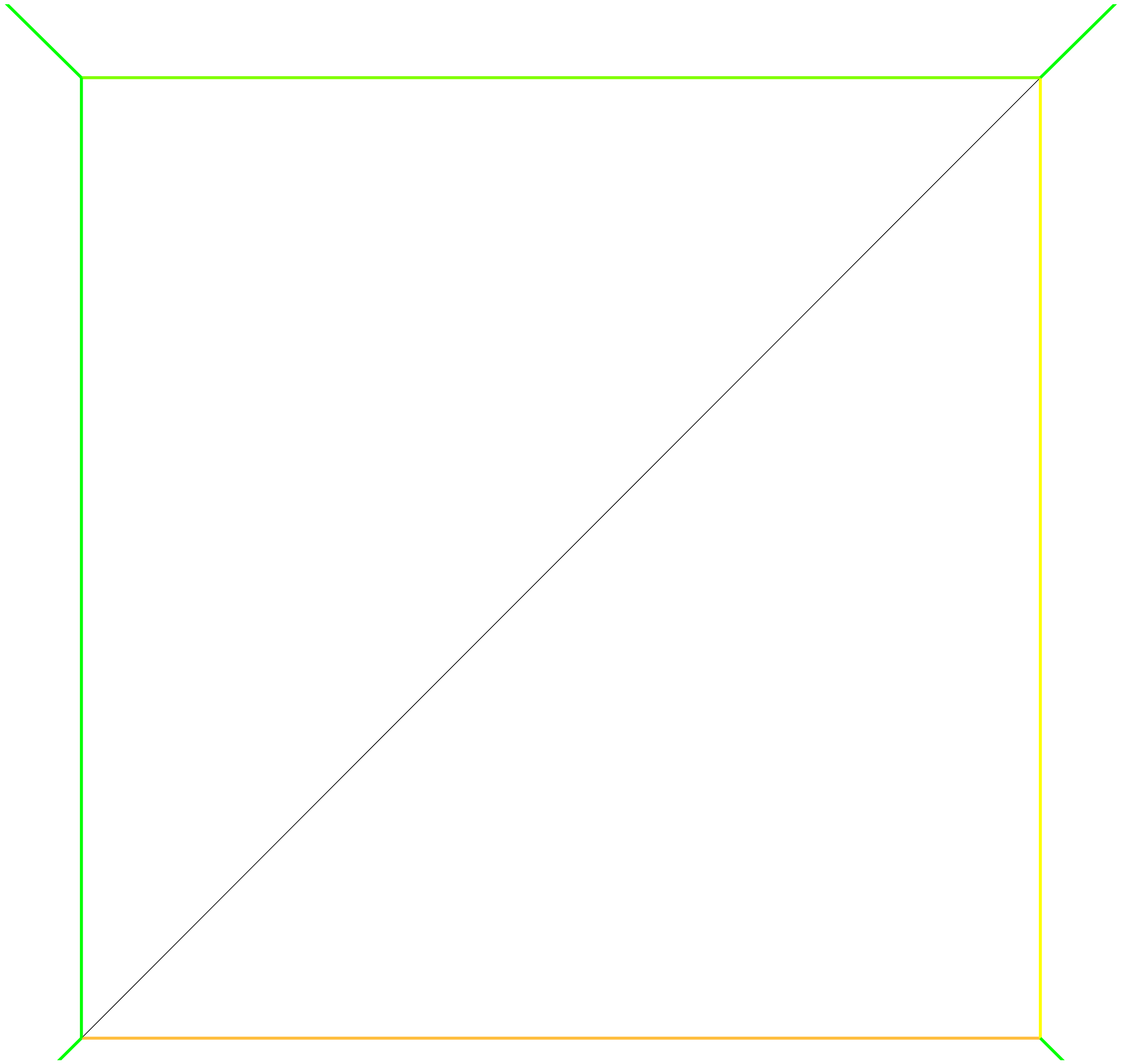

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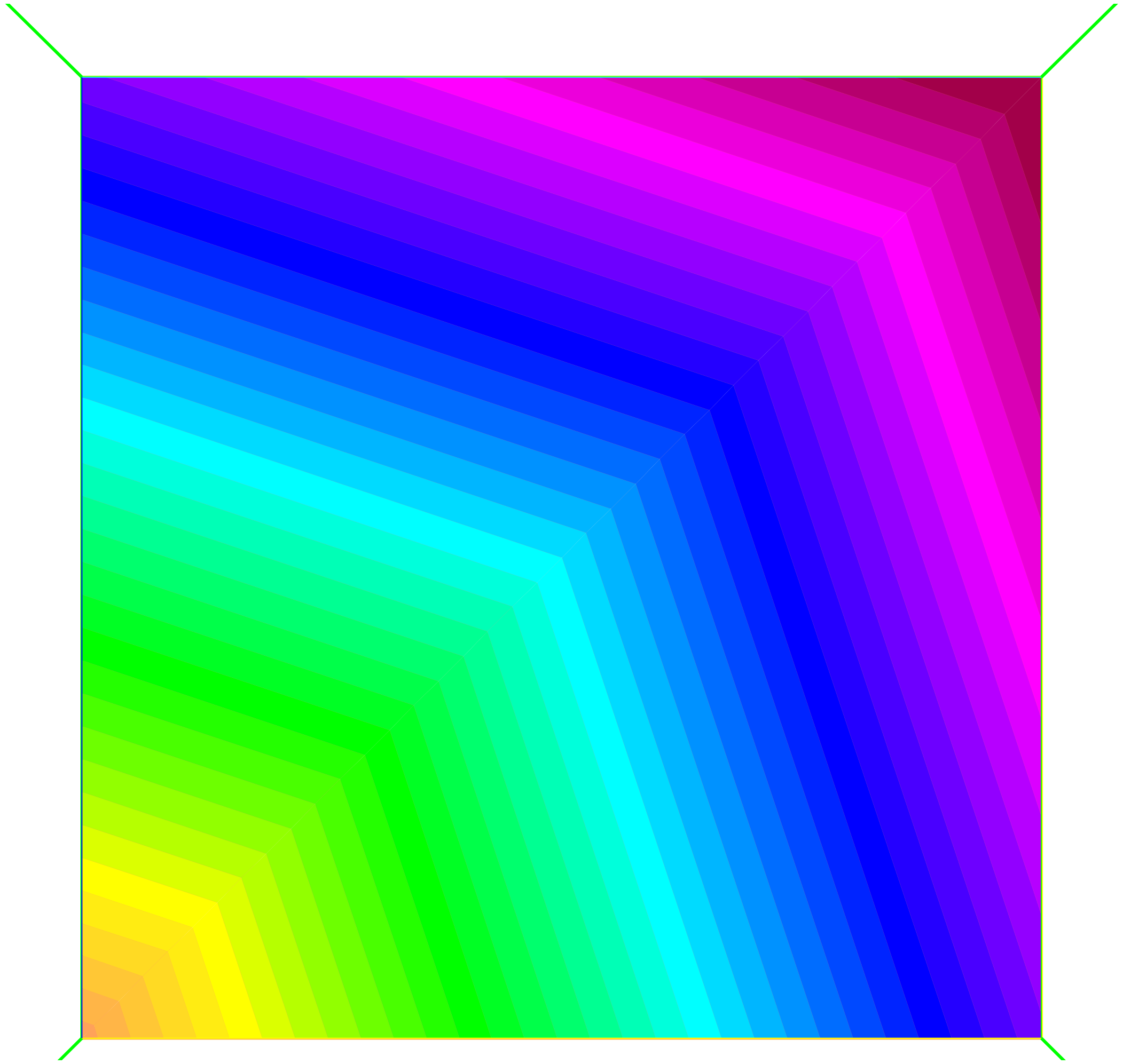

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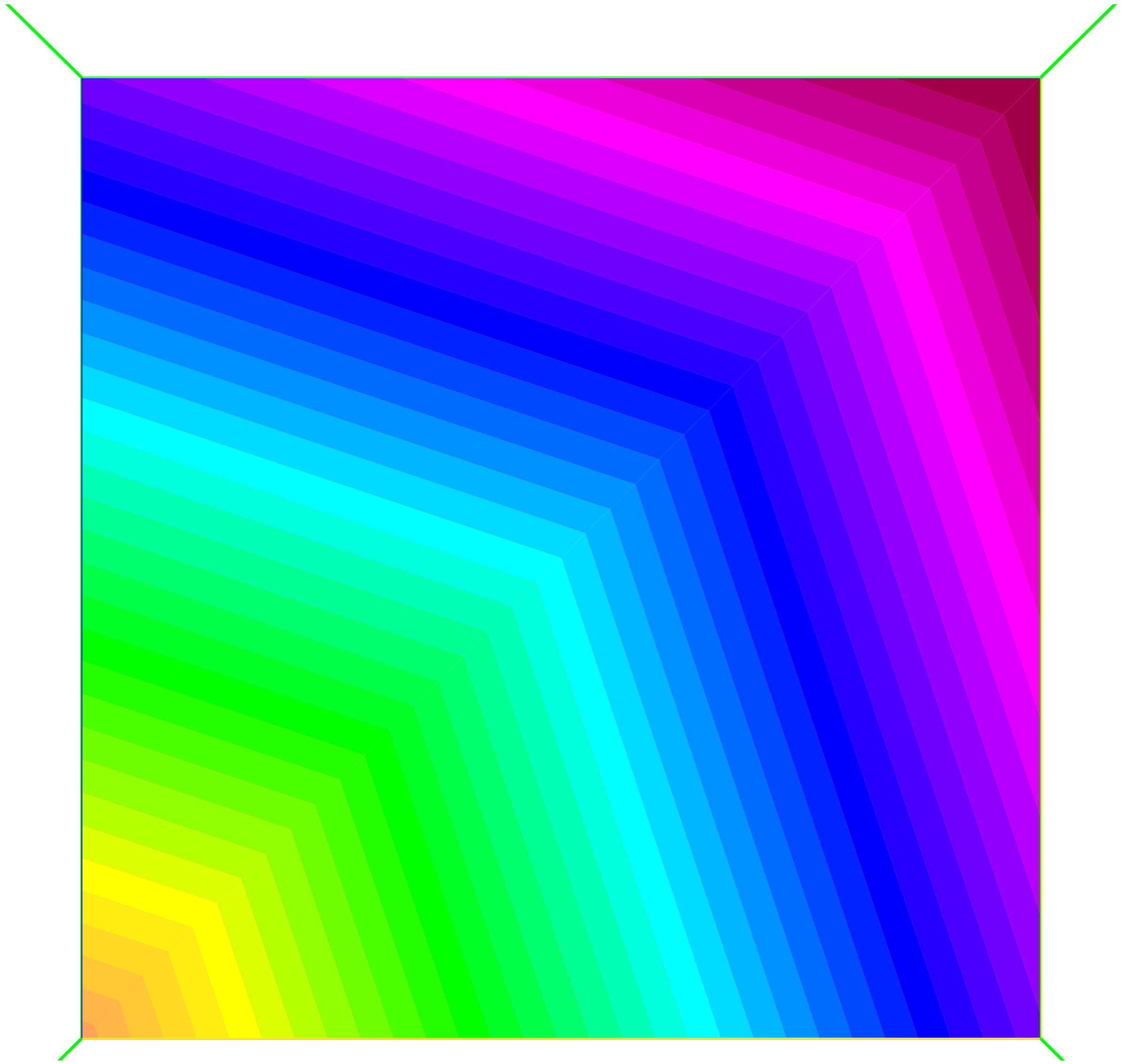

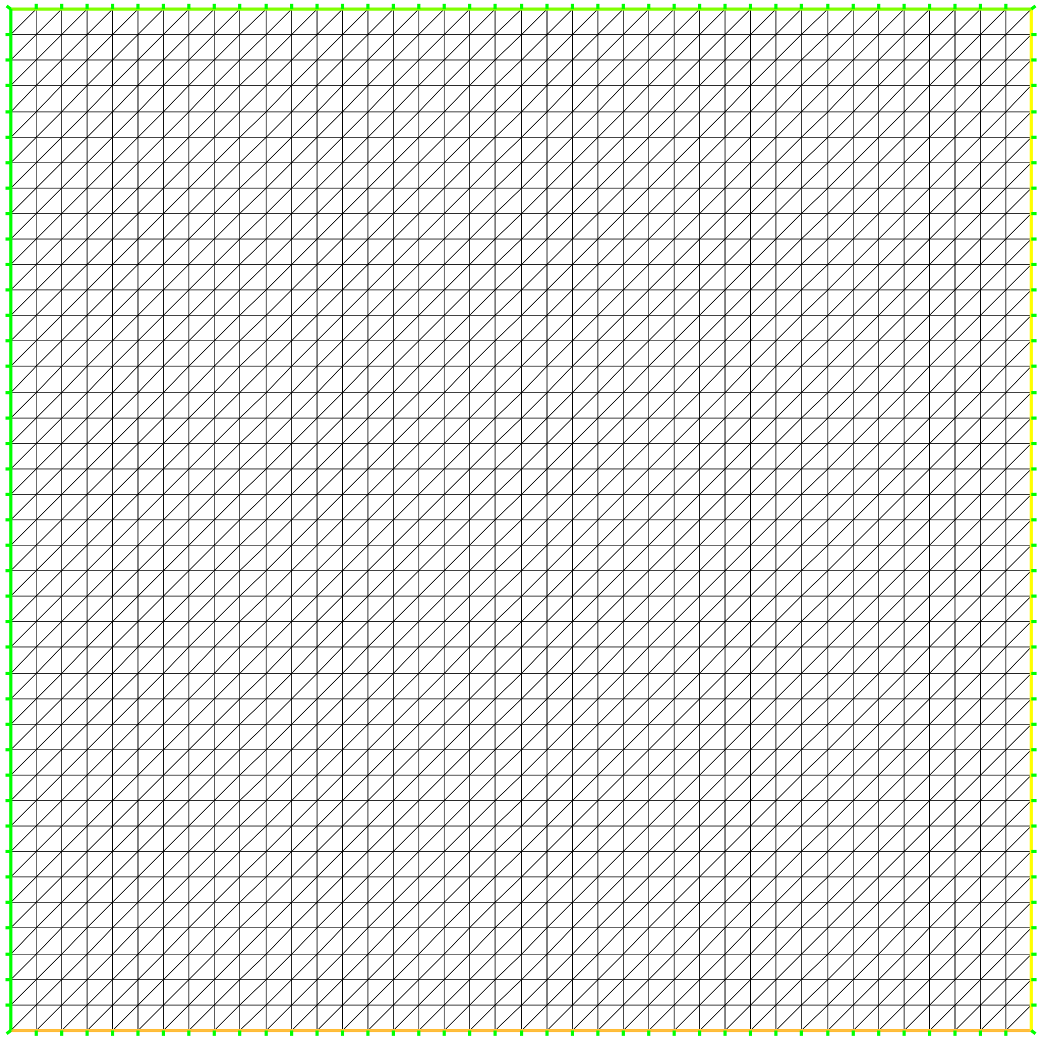

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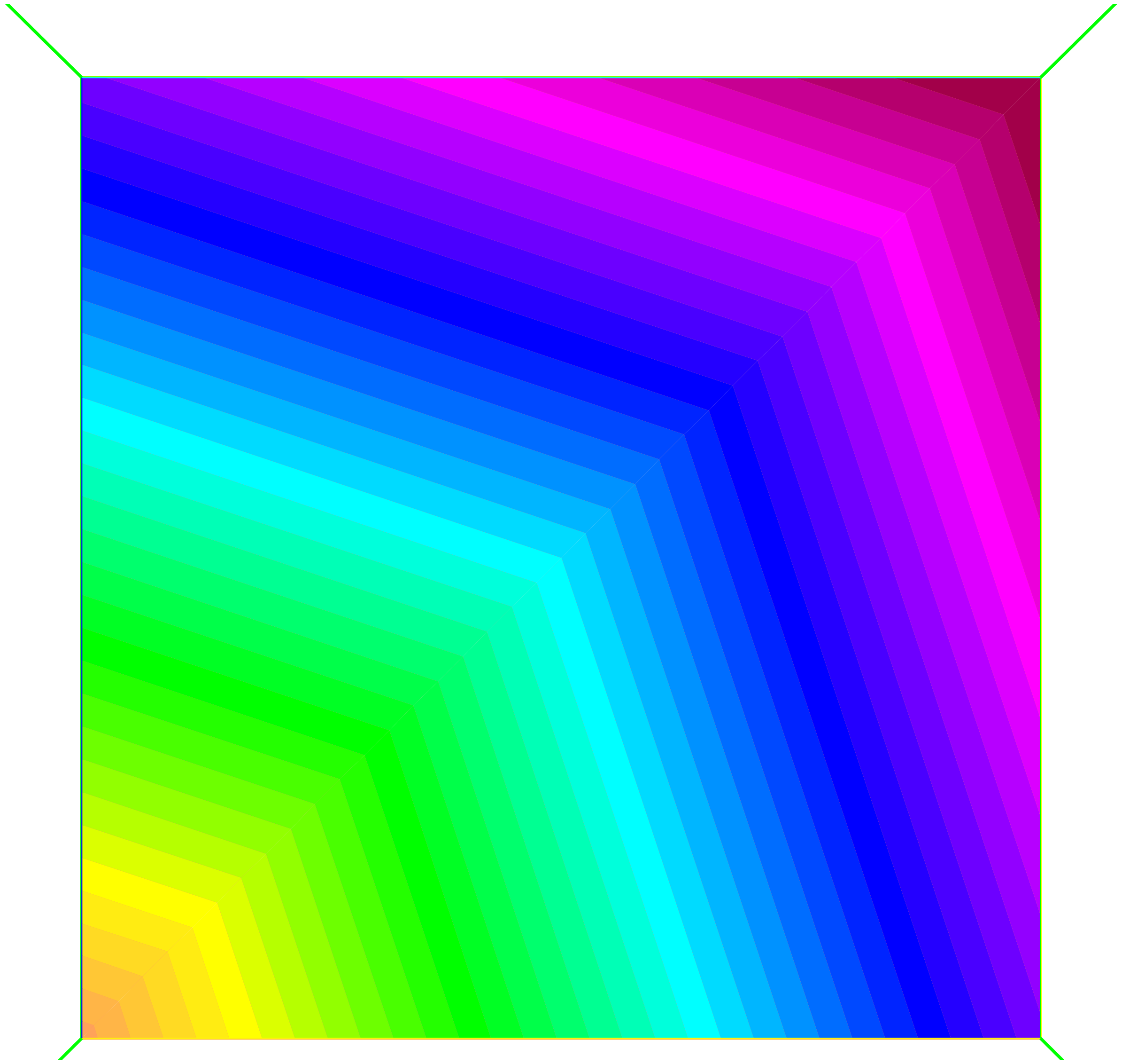

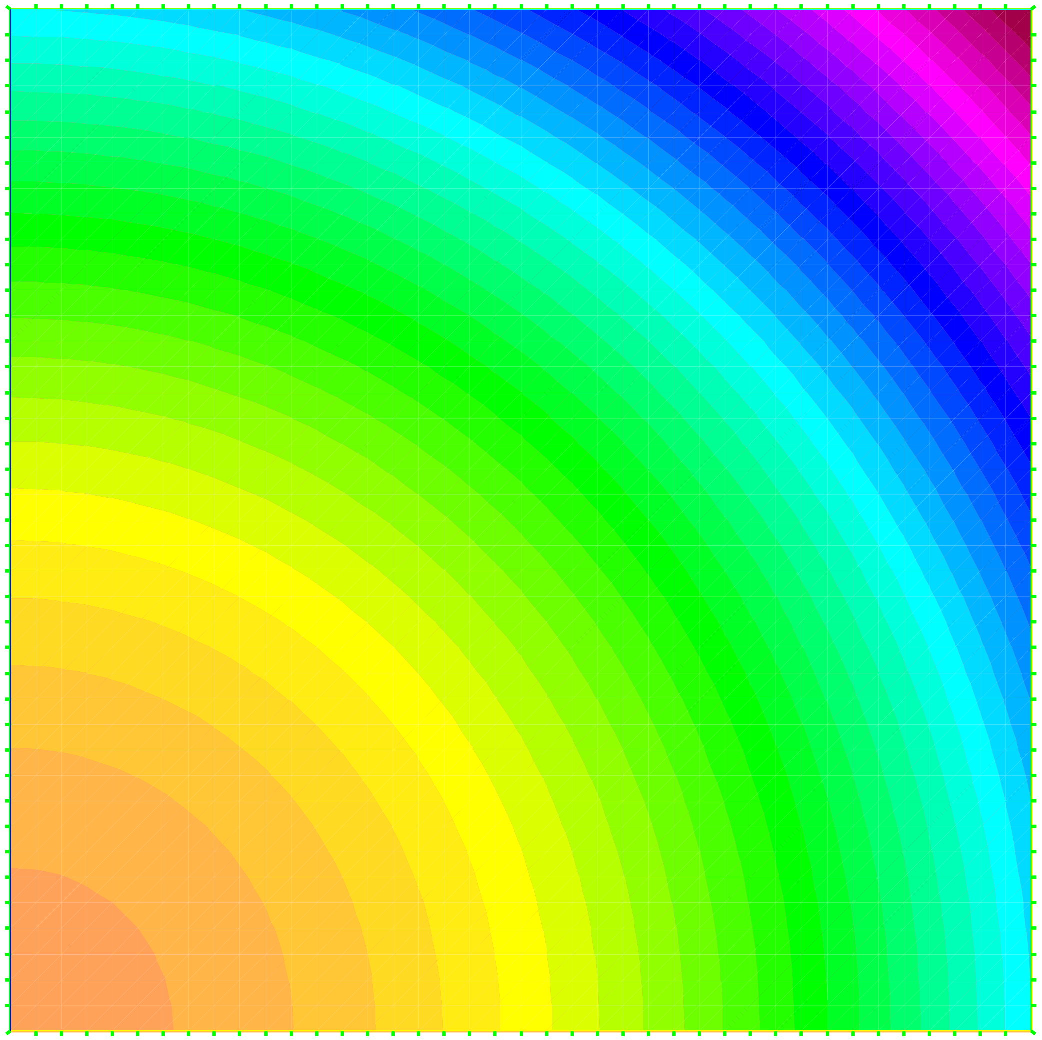

Comparison: conforming and non-conforming

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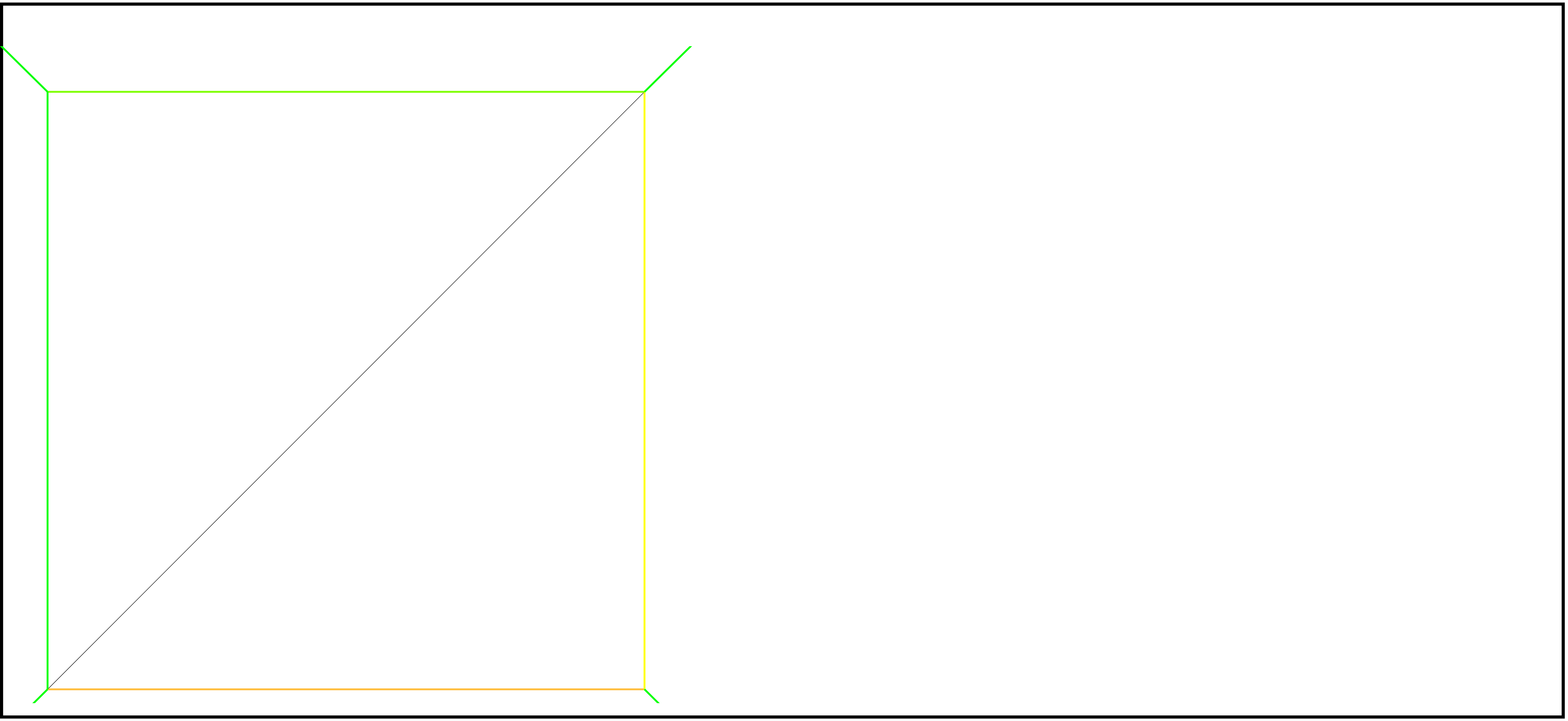

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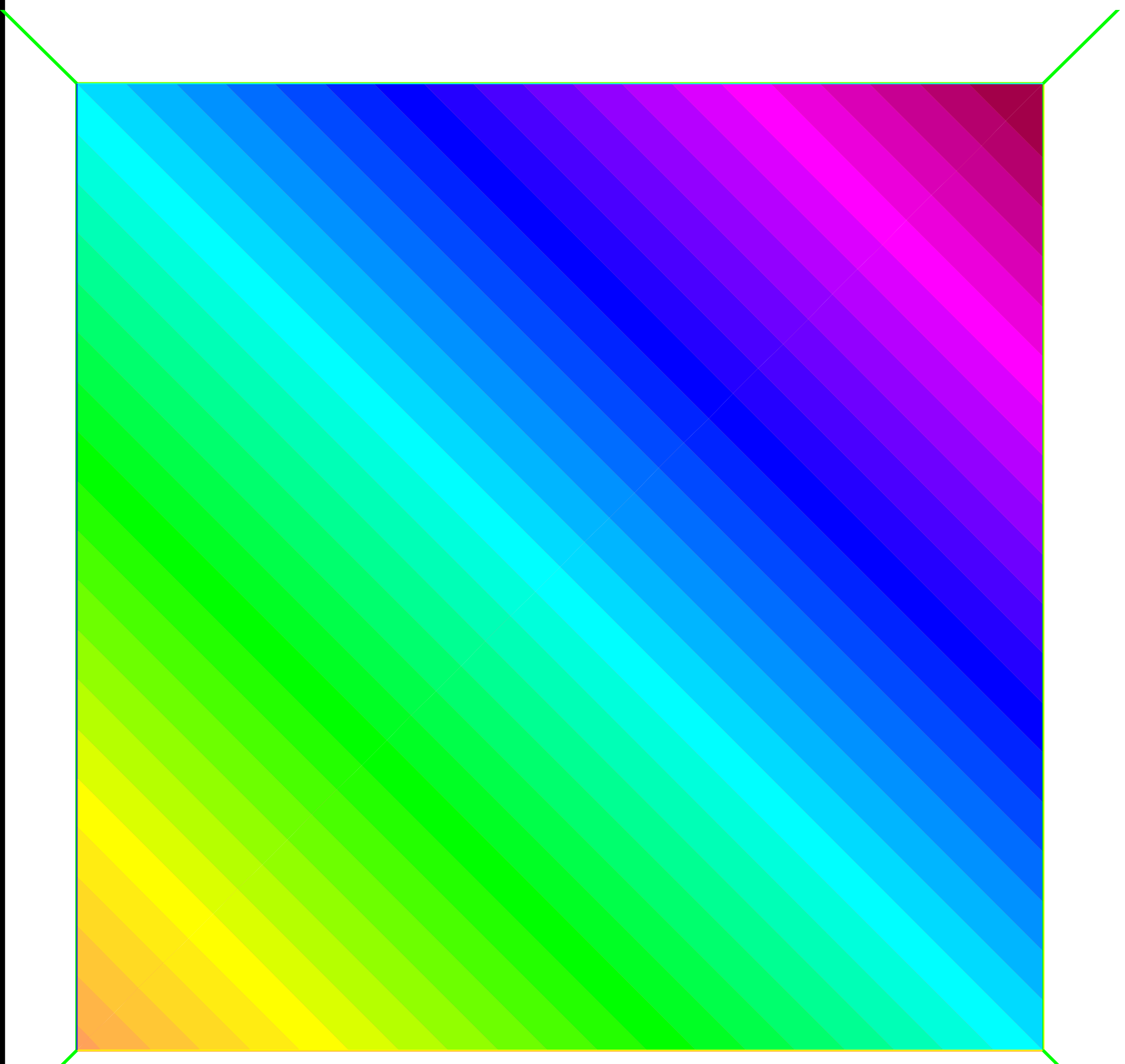

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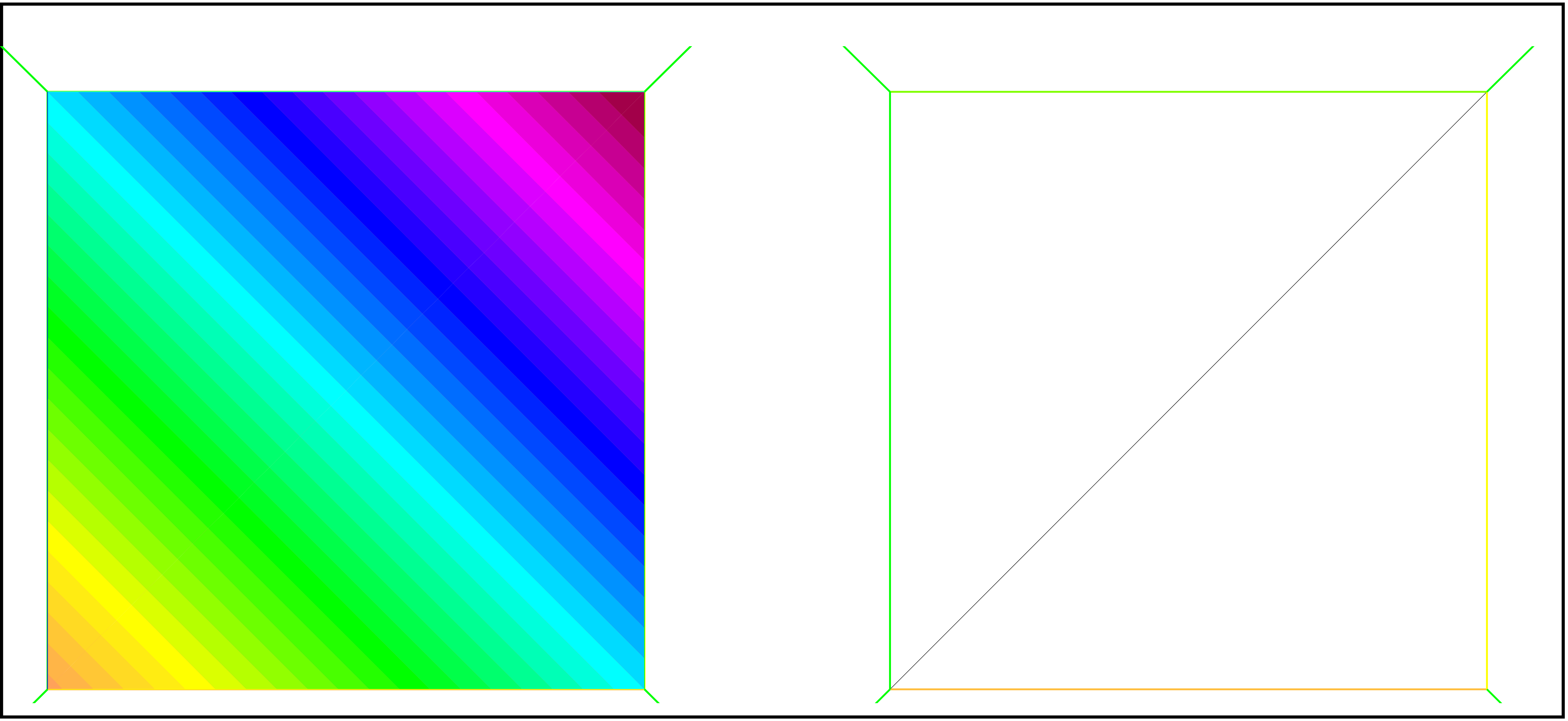

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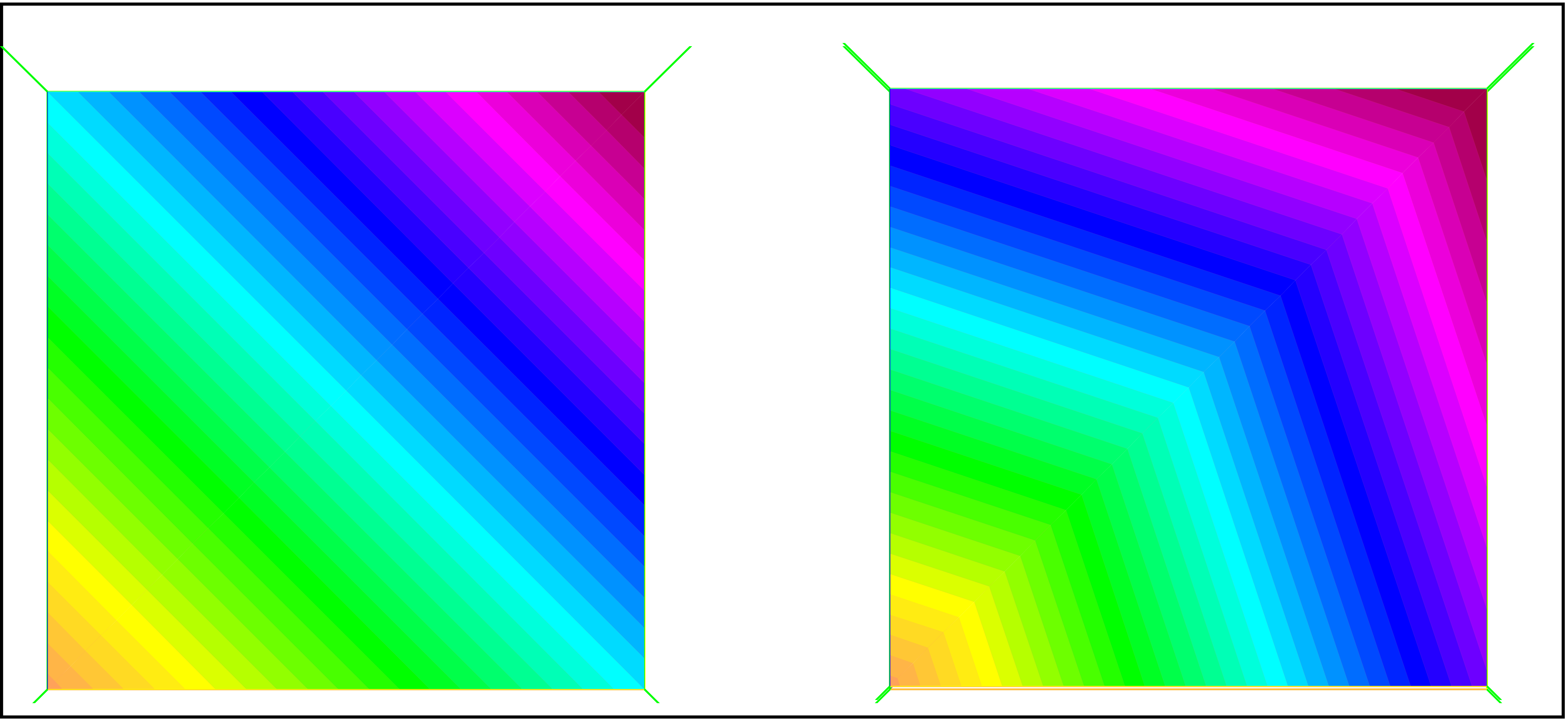

Linear nonconforming elements: best approximation

Best approximation.

Let h be the mesh size of a quasi-regular triangulation. If $w \in H^2(\Omega)$ then $\exists c > 0$ so that

$$\|I_N(u) - u\|_{L^2(\Omega)} \leq ch^m \|u\|_{H^m}, \qquad m = 1, 2.$$

[Armentano and Durán, 2004]

Non-conforming FEM and approximating λ from below

Consider

Model eigenvalue problem, discrete form

Find $(u_N, \lambda_{N,n}) \in (V_N, \mathbb{R})$ such that for all $v_N \in V_N$,

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$$(\nabla u_N, \nabla v_N) = \lambda_{N,n}(u_N, v_N).$$

Is $\lambda_{N,n} \leq \lambda$?

Linear nonconforming elements

Theorem for linear non-conforming elements.

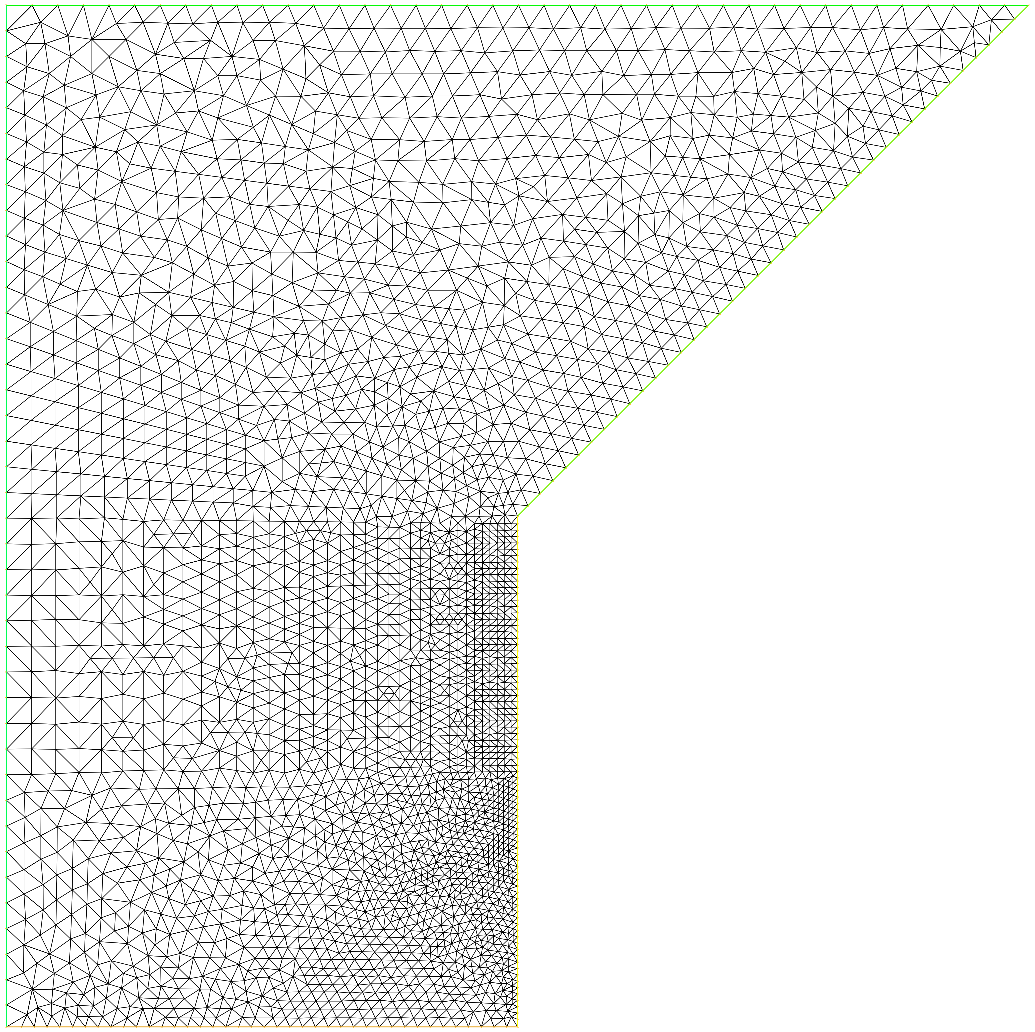
If $(u, \lambda) \in B_2^{1+r,\infty}(\Omega) \times \mathbb{R}$ and if $\exists c > 0$ so that $\|u_N - u\|_h \geq ch^r$ for $r < 1$ then for $h > 0$ small enough

$$\lambda_{N,n} \leq \lambda.$$

[Armentano and Durán, 2004]

This is an *asymptotic* result for *singular* Dirichlet eigenfunctions.

Concrete example:
lowest Dirichlet e.v.



Recap:

- Write original EVP in variational form
- Use conforming or non-conforming finite elements to write discrete EVP
- Have theorems to obtain bounds $\lambda_{N,n} \leq \lambda \leq \lambda_{N,c}$
- For fixed N, approximate discrete eigenvalues $\tilde{\Lambda}_{N,n}, \tilde{\Lambda}_{N,c}$
- Have theorems to estimate $\tilde{\Lambda}_N - \Lambda_N$
- Obtain good estimates for λ.

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Are we done?

Recap:

- Write original EVP in variational form
- Use conforming or non-conforming finite elements to write discrete EVP
- Have theorems to obtain bounds $\Lambda_{N,n} \leq \lambda \leq \Lambda_{N,c}$
- For fixed $h > 0+$, approximate discrete eigenvalues $\tilde{\Lambda}_{N,c}, \tilde{\Lambda}_{N,n}$
- Have theorems to estimate $\Lambda_N - \tilde{\Lambda}_N$ in exact arithmetic!
- Use interval arithmetic to estimate $|\tilde{\ell} - \tilde{\Lambda}_N|$
- Combine to get bounding intervals for λ.

Concretely: Dirichlet evp

$$Err_N := \int_\Omega \|\nabla u_N\|^2 \, dV - \lambda_N \int_\Omega \frac{4|u_N|^2}{(1+r^2)^2} \, dV$$

P1 conforming elements

$\lambda_{c,h}$	Err	N
10.95	1.35135e-15	207
10.7602	1.92541e-14	768
10.7115	5.18084e-14	2978
10.6988	-4.31549e-14	11748
10.6957	-5.93197e-13	46224

P1 non-conforming elements

$\lambda_{nc,h}$	Err	N
10.5863	-1.25056e-14	558
10.664	3.22682e-14	2181
10.6861	2.67427e-14	8691
10.6921	-1.05844e-12	34761
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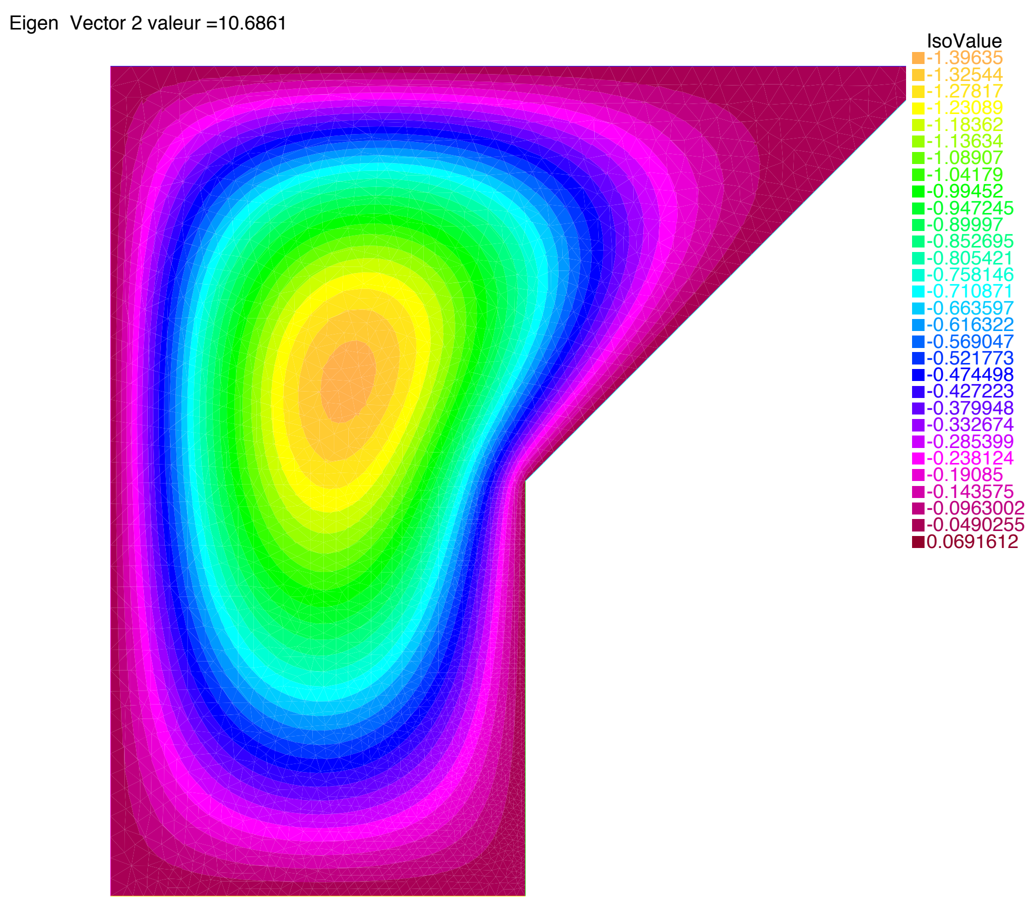

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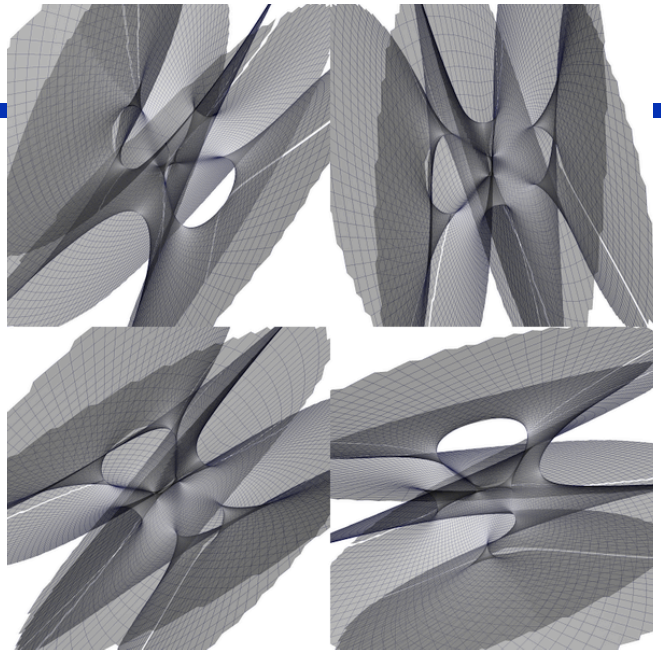
Conjecture:
$\lambda > 10.69$

Spectral problem 2



Jakobson, Levitin, Nadirashvili and Polterovich (preprint '04, JCAM 2006)

- Consider the Bolza surface, $\gamma = 2$, orientable.
- Let $\mathcal{P} := \left\{(z, w) \in \mathbb{C}^2 : w^2 = F(z) = z\frac{(z-1)(z-i)}{(z+1)(z+i)}\right\}$
- $\mathcal{P}$ has the conformal structure of the Bolza surface.
- Let g be the pullback of the round metric $\frac{4dzd\overline{z}}{(1+|z|^2)}$ to $\mathcal{P}$.

Jakobson, Levitin, Nadirashvili and Polterovich, *preprint 2004, JCAM 2006*

Conjecture 1

$$\lambda_1(\mathcal{P})\ Area(\mathcal{P}, g) = 16\pi$$

To show:

$$\lambda_1(\mathcal{P})\ Area(\mathcal{P}, g) = 16\pi$$

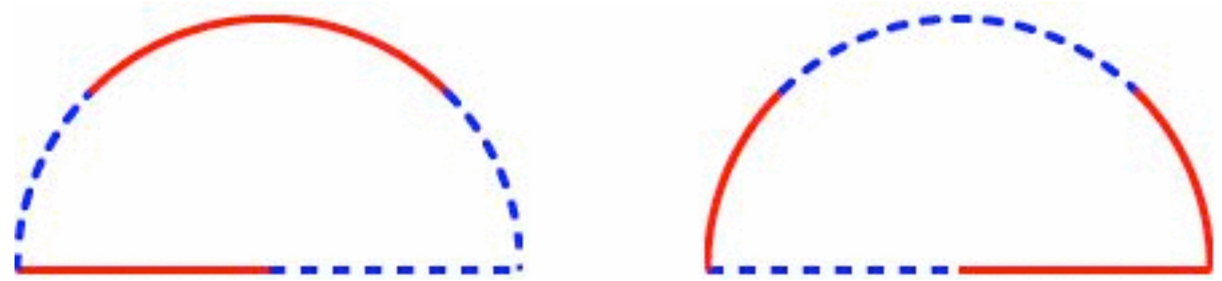
- Let $\Pi : \mathcal{P} \to \mathbb{S}^2$ be a branched covering of second degree, with 6 ramification points.
- The metric g has singularities at these points.
- Define $\lambda_1 := \inf_{u \in H_0^1(\mathcal{P})} \frac{\|\nabla u\|^2}{\|u\|^2}$
- $Area\,(\mathcal{P}, g) = 2\,Area\,(\mathbb{S}^2) = 8\pi.$

Show:

$$\lambda_1(\mathcal{P}) = 2$$

Cannot do this directly!

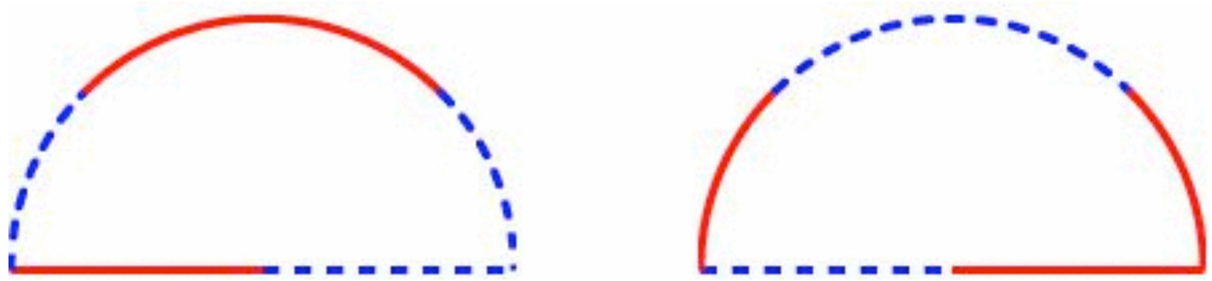
Spectral problem 2



- Let Ω be the half-disk in $\mathbb{R}^2$
- Prescribe Dirichlet data on solid segments, Neumann on rest
- Find first eigenvalue of

$$-\Delta u = \lambda \frac{4}{(1+r^2)^2} u \text{ in } \Omega,$$

Spectral problem 2



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Conjecture 2

$$\lambda > 2.$$

Theorem

Conjecture 2 implies Conjecture 1.

Use numerics to assist?

This is still hard....use careful numerical simulation.

Find $\lambda \in \mathbb{R}, u \in H^1_{\partial D}(\Omega)$ so

$$-\Delta u = \lambda \frac{4}{(1+r^2)^2} u \, in \, \Omega,$$

with mixed Dirichlet-Neumann data.
Generalized eigenvalue problem: $\lambda_N \in \mathbb{R}, u_N \in V_N$

$$\int_\Omega \nabla u_N \cdot v_N \, dV = \lambda_N 4 \int_\Omega \frac{u_N v_N}{(1+r^2)^2} \, dV$$

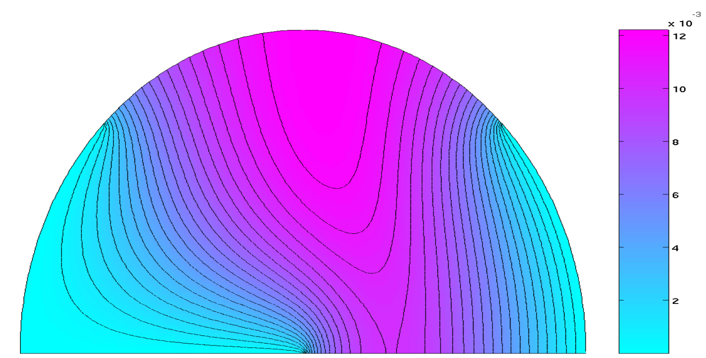
- True solution is not available.
- Need to conclude, after a fixed number of refinements that $\lambda_N > 2 \Rightarrow \lambda > 2$.

Numerical results

$$Err_N := \int_\Omega \|\nabla u_N\|^2 \, dV - \lambda_N \int_\Omega \frac{4|u_N|^2}{(1+r^2)^2} \, dV$$

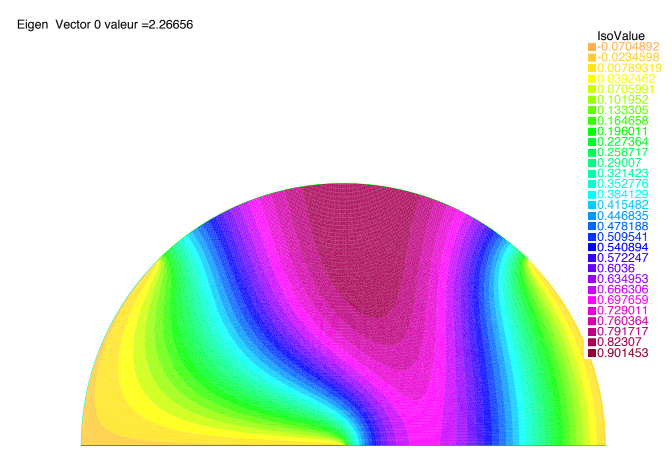
P1 conforming elements

$\lambda_{c,h}$	Err	N
2.42641	-2.32063e-15	253
2.34502	-7.59321e-16	965
2.31024	-1.20319e-14	3733
2.29249	-2.32825e-13	14880
2.28383	-1.76215e-12	58563



P1 non-conforming elements

$\lambda_{nc,h}$	Err	N
2.14419	-1.46411e-15	696
2.20874	8.27631e-15	2772
2.24127	-1.16138e-13	10956
2.25803	-1.02808e-12	44157
2.26656	-1.58561e-11	174726



Observation

$$\lambda_{N,n} \searrow, \qquad \lambda_{N,c} \nearrow$$

Observation

$$\lambda_{N,n} \searrow, \qquad \lambda_{N,c} \nearrow$$

Conclude $\lambda > 2$?

Numerical evidence. [Jakobson, Levitin, Nadirishvili, NN, Polterovich '05.]

Recap:

- Write original EVP in variational form
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- Have theorems to obtain bounds $\lambda_{nch} \leq \lambda \leq \lambda_{ch}$ No
- For fixed $h > 0+$, approximate discrete eigenvalues $\tilde{\lambda}_{nch}, \tilde{\lambda}_{ch}$
- Have theorems to estimate $\lambda_h - \tilde{\lambda}_h$ in exact arithmetic!

Closer to a proof.

- Theorem

Let $h > 0$ be sufficiently small. Then the discrete eigenvalues using non-conforming linear FEM approximate λ from below.

- Derivatives of eigenfunction are singular. Eg, near origin,

$$w_{j,n}(r,\theta) = r^b(1+r^2)^{\frac{1-\sqrt{1+\lambda}}{2}}\ {}_2F_1(\tilde{\alpha},\tilde{\alpha}+1;1+b;-r^2)\sin(\frac{j\pi}{2\alpha}\theta)$$

- Discrete eigenvalues using non-conforming Crouziex-Raviert elements approximate the true eigenvalue from below.

Recap:

- Write original EVP in variational form
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- Have theorems to obtain bounds $\lambda_{nch} \leq \lambda \leq \lambda_{ch}$
- For fixed $h > 0+$, approximate discrete eigenvalues $\tilde{\lambda}_{nch}, \tilde{\lambda}_{ch}$
- Have theorems to estimate $\lambda_h - \tilde{\lambda}_h$ in exact arithmetic!
- Use interval arithmetic to estimate

Perhaps what we could do.

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Conjecture

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Conjecture → Proof

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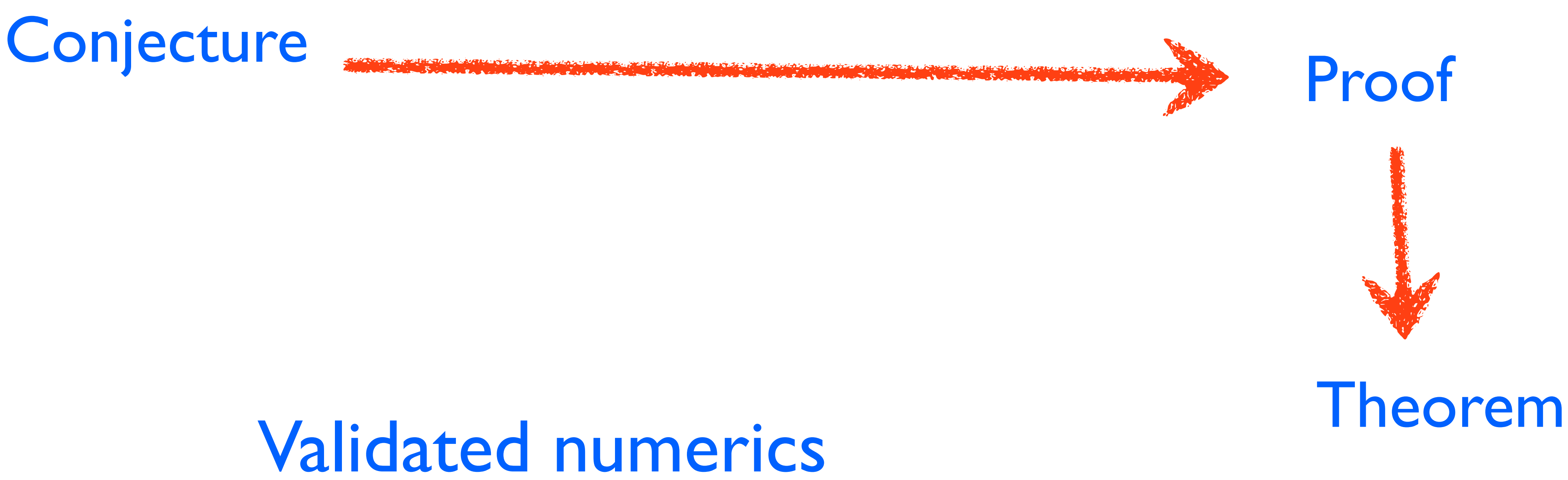

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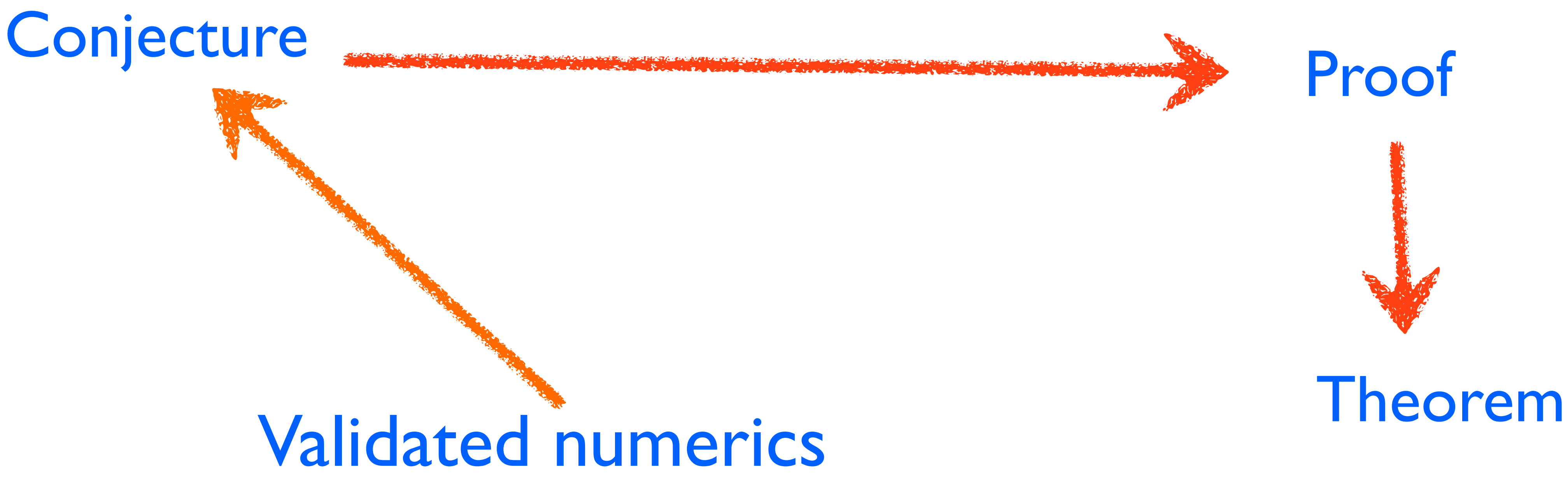

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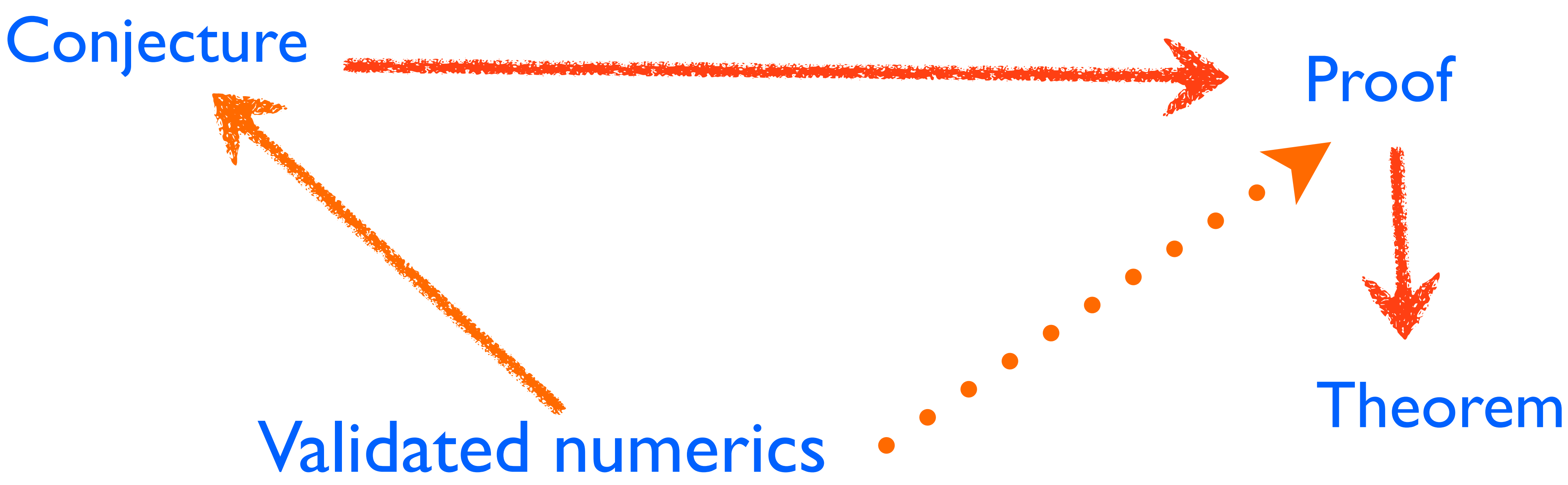

Thanks to

Thanks to the organizers

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NSERC CRSNG

MITACS

Fonds de recherche sur la nature et les technologies Québec

Cross check

- Fix the mesh
- Let $(\lambda_{N,n}, u_{N,n})$ be a non-conforming approximation
- interpolate $u_{N,n}$ by *conforming* elements on same mesh
- Check residual

$$Err_{n\to c} \int_{\Omega} |\nabla \mathcal{I}_{n\to c} u_{N,n}|^2 - \lambda_{N,n} \frac{4}{(1+r^2)^2} |\mathcal{I}_{n\to c} u_{N,n}\|^2 \, dA$$

Cross-checking residuals

$Err_{n\to c}$	N_{dof}	$Err_{c\to n}$	N_{dof}
-0.295619	253	0.282211	696
0.0484749	965	0.13628	2772
0.0106499	3733	0.0689692	10956
0.00450842	14880	0.0344616	44157
-0.00149719	58563	0.0172685	174726