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Nonlinear flows and rigidity results on compact manifolds

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Joint work with Jean Dolbeault and Maria Esteban

$(\mathcal{M}, g)$ a compact d-dimensional Riemannian manifold without boundary.

Can assume that

$$|\mathcal{M}| = \int_{\mathcal{M}} d\text{vol} = 1 \ .$$

Consider

$$-\Delta v + \frac{\lambda}{p-2}(v - v^{p-1}) = 0 \quad (*)$$

λ some constant, $2 < p < \frac{2d}{d-2}$

The function 1 is a solution

Is it the only one?

The answer depends on the value of the constant λ

A minimizer of the functional

$$\frac{\int_{\mathcal{M}} |\nabla v|^2 + \frac{\lambda}{p-2} \int_{\mathcal{M}} v^2}{\left(\int_{\mathcal{M}} v^p\right)^{p/2}} \quad (**)$$

is a positive solution of $(*)$ but not conversely

For λ small one expects that the constants are the only minimizers

Let λ_1 be the lowest non-vanishing eigenvalue of $-\Delta$ on $\mathcal{M}$

If $\lambda > \lambda_1$ then the minimizer is not a constant function

and hence the equation $(*)$ has non-constant solutions

Bochner - Lichnerowicz - Weizenböck formula

$$\frac{1}{2}\Delta\,|\nabla u|^2 = \|\mathrm{H}_g u\|^2 + \nabla(\Delta_g u)\cdot\nabla u + \mathrm{Ric}(\nabla u, \nabla u)$$

$$\mathrm{Ric}(\cdot,\cdot)$$

is the Ricci tensor

If $\mathcal{M} = \mathbb{S}^d$ then

$$\mathrm{Ric}(\cdot,\cdot) = (d-1)g(\cdot,\cdot)$$

Define

$$\theta = \frac{(d-1)^2\,(p-1)}{d\,(d+2)+p-1} \text{ and}$$

$$\lambda_\star := \inf_{u\in H^2(\mathcal{M})} \frac{(1-\theta)\displaystyle\int_{\mathcal{M}} (\Delta u)^2 + \frac{\theta\,d}{d-1}\int_{\mathcal{M}} \mathrm{Ric}(\nabla u,\nabla u)}{\int_{\mathcal{M}} |\nabla u|^2} .$$

Note that $0<\theta<1$ since $1<p<\frac{2d}{d-2}$

Theorem: (Dolbeault, Esteban, L) If

$$0<\lambda<\lambda_\star$$

then for any $2<p<\frac{2d}{d-2}$, equation $(*)$ has 1 as the only solution.

Corollary:

$$\text{Let} \quad \rho = \inf_{\eta \in \mathbb{S}^d} \operatorname{Ric}(\eta, \eta) > 0$$

and let λ_1 be the first non-vanishing eigenvalue of $-\Delta$.

If $\lambda \leq (1-\theta)\lambda_1 + \theta \frac{d\rho}{d-1}$ then $(*)$ has 1 as the only solution.

This follows from the inequality

$$\int_{\mathcal{M}} (\Delta u)^2 \geq \lambda_1 \int_{\mathcal{M}} |\nabla u|^2$$

Equality holds precisely when $u = \text{const.} + \phi$ where ϕ is an eigenfunction of the first non-zero eigenvalue of $-\Delta$.

$$\mathcal{M} = \mathbb{S}^d$$

$$\lambda_1 = d \ , \rho = (d-1)$$

and hence for all $\lambda \leq d$, $(*)$ has 1 as the only solution.

Corollary:

$$\int_{\mathbb{S}^d} |\nabla v|^2 d\sigma \geq \frac{\lambda}{p-2} \left[\|v\|^2_{L^p(\mathbb{S}^d)} - \|v\|^2_{L^2(\mathbb{S}^d)} \right]$$

for all $\lambda \leq d$ and $p < \frac{2d}{d-2}$

with equality if and only if v is the constant function

History:

Obata 1971/72, Gidas and Spruck 1981

1990 Bidaut-Véron and Véron , Beckner 1991,

Licois and Véron 1995, Bakry and Ledoux 1996, Demange 2008

The main idea is similar to the one used by Demange

by constructing a flow that transports any function to an optimizer

Sketch of the proof:

$$v = u^{\beta}\,, \quad \beta \text{ will be chosen later}$$

$$\mathcal{F}[u] := \int_{\mathcal{M}} |\nabla(u^{\beta})|^2 + \frac{\lambda}{p-2}\left[\int_{\mathcal{M}} u^{2\,\beta} - \left(\int_{\mathcal{M}} u^{\beta\,p}\right)^{2/p}\right].$$

$$\text{Flow}: \; u_t = u^{2-2\,\beta}\{\Delta u + \kappa\,\frac{|\nabla u|^2}{u}\}$$

a porous medium type flow

$$\kappa = \beta(p-2)+1$$

so that

$$\frac{d}{dt}\int_{\mathcal{M}} u^{p\beta} dv_g = 0$$

and

$$\frac{1}{2\,\beta^2}\,\frac{d}{dt}\mathcal{F}[u] = -\int_{\mathcal{M}}\left[(\Delta_g u)^2 + (\kappa+\beta-1)\,\Delta_g u\,\frac{|\nabla u|^2}{u} + \kappa\,(\beta-1)\,\frac{|\nabla u|^4}{u^2}\right]$$

$$+\lambda\int_{\mathcal{M}} |\nabla u|^2\,.$$

$$\frac{1}{2\,\beta^2}\,\frac{d}{dt}\mathcal{F}[u] = -\,(1-\theta)\int_{\mathcal{M}}(\Delta_g u)^2 - \mathcal{G}[u] + \lambda\int_{\mathcal{M}}|\nabla u|^2\,,$$

where

$$\mathcal{G}[u] := \int_{\mathcal{M}}\left[\theta\,(\Delta_g u)^2 + (\kappa+\beta-1)\,\Delta_g u\,\frac{|\nabla u|^2}{u} + \kappa\,(\beta-1)\,\frac{|\nabla u|^4}{u^2}\right].$$

To understand $\mathcal{G}(u)$ the following quantity is key

$$\mathrm{Q}_g^\theta u := \mathrm{L}_g u - \frac{1}{\theta}\,\frac{d-1}{d+2}\,(\kappa+\beta-1)\left[\frac{\nabla u\otimes\nabla u}{u} - \frac{g}{d}\,\frac{|\nabla u|^2}{u}\right].$$

$$\mathrm{L}_g u := \mathrm{H}_g u - \frac{g}{d}\,\Delta_g u\,,\ \text{trace free Hessian}$$

Theorem

Assume that $d \geq 2$. With the above notations, any positive function $u \in C^2(\mathcal{M})$ satisfies the identity

$$\mathcal{G}[u] = \frac{\theta\, d}{d-1} \left[\int_{\mathcal{M}} \|\mathrm{Q}_g^\theta u\|^2 + \int_{\mathcal{M}} \mathrm{Ric}(\nabla u, \nabla u) \right] - \mu \int_{\mathcal{M}} \frac{|\nabla u|^4}{u^2}$$

with $\mu := \frac{1}{\theta}\big(\frac{d-1}{d+2}\big)^2(\kappa+\beta-1)^2 - \kappa\,(\beta-1) - (\kappa+\beta-1)\,\frac{d}{d+2}$ so that

$$\frac{1}{2\,\beta^2}\,\frac{d}{dt}\mathcal{F}[u] = -\,(1-\theta) \int_{\mathcal{M}} (\Delta_g u)^2 - \frac{\theta\, d}{d-1} \left[\int_{\mathcal{M}} \|\mathrm{Q}_g^\theta u\|^2 + \int_{\mathcal{M}} \mathrm{Ric}(\nabla u, \nabla u) \right] + \mu \int_{\mathcal{M}} \frac{|\nabla u|^4}{u^2} + \lambda \int_{\mathcal{M}} |\nabla u|^2$$

Lemma 1:

$$\int_{\mathcal{M}} (\Delta_g u)^2 = \frac{d}{d-1} \int_{\mathcal{M}} \| \, \mathrm{L}_g u \, \|^2 + \frac{d}{d-1} \int_{\mathcal{M}} \mathrm{Ric}(\nabla u, \nabla u).$$

$$\mathrm{A} \cdot \mathrm{B} := g^{i,m} \, g^{j,n} \, \mathrm{A}_{i,j} \, \mathrm{B}_{m,n} \quad \text{and} \quad \|\mathrm{A}\|^2 := \mathrm{A} \cdot \mathrm{A} \; .$$

$$\mathrm{L}_g u := \mathrm{H}_g u - \frac{g}{d} \Delta_g u \;, \; \text{trace free Hessian}$$

Proof:

$$\frac{1}{2}\Delta\left|\nabla u\right|^2 = \|\mathrm{H}_g u\|^2 + \nabla(\Delta_g u)\cdot\nabla u + \mathrm{Ric}(\nabla u, \nabla u)$$

$$\int_{\mathcal{M}} (\Delta_g u)^2 = \int_{\mathcal{M}} \|\mathrm{H}_g u\|^2 + \int_{\mathcal{M}} \mathrm{Ric}(\nabla u, \nabla u)$$

$$\int_{\mathcal{M}} \|\,\mathrm{L}_g u\,\|^2 = \int_{\mathcal{M}} \|\mathrm{H}_g u\|^2 - \frac{1}{d}\int_{\mathcal{M}} (\Delta_g u)^2,$$

Lemma 2:

$$\int_{\mathcal{M}} \Delta_g u \,\frac{|\nabla u|^2}{u} = \frac{d}{d+2}\int_{\mathcal{M}} \frac{|\nabla u|^4}{u^2} - \frac{2\,d}{d+2}\int_{\mathcal{M}} [\mathrm{L}_g u]\cdot\left[\frac{\nabla u \otimes \nabla u}{u}\right]$$

Proof:

$$\int_{\mathcal{M}} \Delta_g u \,\frac{|\nabla u|^2}{u} = \int_{\mathcal{M}} \frac{|\nabla u|^4}{u^2} - 2\int_{\mathcal{M}} [\mathrm{H}_g u]\cdot\left[\frac{\nabla u \otimes \nabla u}{u}\right]$$

$$\mu = \left(\frac{1}{\theta} \left(\frac{d-1}{d+2} \right)^2 (p-1)^2 - (p-2) \right) \beta^2 - 2\, \frac{d+3-p}{d+2}\, \beta + 1$$

$$\beta = \frac{(d+2)\,(d+3-p)\,\theta}{(d-1)^2\,(p-1)^2 - (d+2)^2\,(p-2)\,\theta}$$

$$\theta = \frac{(d-1)^2\,(p-1)}{d\,(d+2)+p-1} \quad \text{and} \quad \beta = \frac{d+2}{d+3-p}$$

so that

$$\mathcal{G}[u] = \frac{\theta\, d}{d-1} \left[\int_{\mathcal{M}} \|\mathrm{Q}_g^\theta u\|^2 + \int_{\mathcal{M}} \mathrm{Ric}(\nabla u, \nabla u) \right]$$

Two points of view:

Just a trial function calculation, nothing is really needed about the flow

Let $v = u^\beta$ be a minimizer of (**). Then consider

$$u \longrightarrow u + \varepsilon u^{2-2\,\beta}\{\Delta u + \kappa\,\frac{|\nabla u|^2}{u}\}$$

Since

$$\frac{1}{2\,\beta^2}\,\frac{d}{dt}\mathcal{F}[u] \le (\lambda - \lambda_\star)\int_{\mathcal{M}} |\nabla u|^2$$

we conclude that $\nabla u = 0$ for $\lambda < \lambda_\star$

Flow drives any function to a constant function

This shows that

$$0 = \mathcal{F}(\|u^\beta\|_p) \leq \mathcal{F}(u^\beta)$$

Nothing is said about the uniqueness of the optimizer

Example: $p = \frac{2d}{d-2}$, $d \neq 3$

$$\theta = 1 \ , \quad \beta = \frac{d-2}{d-3}$$

$$\frac{1}{2\,\beta^2}\,\frac{d}{dt}\mathcal{F}[u] = \lambda \int_{\mathcal{M}} |\nabla u|^2 - \frac{d}{d-1}\int_{\mathcal{M}} \mathrm{Ric}(\nabla u, \nabla u) - \frac{d}{d-1}\int_{\mathcal{M}} \|Q_g^1 u\|^2$$

$$\Lambda_\star := \inf_{u\in H^2(\mathcal{M})\setminus\{0\}} \frac{(1-\theta)\displaystyle\int_{\mathcal{M}} (\Delta_g u)^2 + \frac{\theta\, d}{d-1}\int_{\mathcal{M}} \Big[\, \|\mathrm{Q}_g^\theta u\|^2 + \mathrm{Ric}(\nabla u, \nabla u)\Big]}{\displaystyle\int_{\mathcal{M}} |\nabla u|^2} \,.$$

Theorem (Dolbeault, Esteban, L)

For any $p \in [1,2)\cup(2,2^*]$ if $d\ge 3$, $p\in(1,2)\cup(2,\infty)$ if $d=1$ or 2, inequality $(**)$ holds for all $\lambda \le \Lambda$ where $\Lambda \in [\Lambda_\star, \lambda_1]$. Moreover, if $\Lambda_\star < \lambda_1$, then Λ is such that

$$\Lambda_\star < \Lambda \le \lambda_1 \,.$$

If $d=1$, then $(**)$ holds for all $\lambda \le \lambda_1$.