Complex spectra of self-adjoint operator pencils

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based on joint works with

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(http://arxiv.org/abs/1303.2185, now in revision)

and

with E Brian Davies (King's College London)
(in preparation)

Self-adjoint pencils

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Thus, the interesting case is when *both* A and B are not sign-definite — the pencil spectrum can be non-real.

Complex eigenvalues and typical questions

Little can be deduced about non-real eigenvalues from the general principles. E.g. variational approach gives for an eigenvalue λ:

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Simple matrix pencil

We consider the following class of problems. Fix an integer $N \in \mathbb{N}$, and define the classes of $N \times N$ matrices $H_{N;c}$ and $D_{m,n;\sigma,\tau}$, where

$$H_{N;c} = \begin{pmatrix} c & 1 & 0 & \dots & 0 \\ 1 & c & 1 & \dots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \dots & 1 & c & 1 \\ 0 & \dots & 0 & 1 & c \end{pmatrix}$$

is tri-diagonal, $c \in \mathbb{R}$ is a parameter, and

Simple matrix pencil (contd.)

$$D_{m,n;\sigma,\tau} = \begin{pmatrix} \sigma & & & & & \\ & \ddots & & & & \\ & & \sigma & & & \\ & & & \tau & & \\ & & & & \ddots & \\ & & & & & \tau \end{pmatrix} \quad \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} m \text{ rows} \\ \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} n \text{ rows} \end{matrix}$$

is diagonal, where $m, n \in \mathbb{N}$ and $\sigma, \tau \in \mathbb{C}$ are parameters, and we assume $m + n = N$.

We are only going to consider the case $\sigma = -\tau = 1$, and denote for brevity

$$D_{m,n} := D_{m,n;1,-1}$$

We study the eigenvalues of the linear operator pencil

$$\mathcal{P}_{m,n;c} = \mathcal{P}_{m,n;c}(\lambda) = H_{m+n;c} - \lambda D_{m,n}$$

as $N = m + n \to \infty$.

Basics

We start with the following easy result on the localisation of eigenvalues of the pencil $\mathcal{P}_{m,n;c}$.

Theorem

(a) *The spectrum* $\operatorname{spec} \mathcal{P}_{m,n;c}$ *is invariant under the symmetry* $\lambda \to \overline{\lambda}$.

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(c) *If* $|c| \geq 2$, *then* $\operatorname{spec} \mathcal{P}_{m,n;c} \subset \mathbb{R}$.

Rough localisation

Rough asymptotics of eigenvalues as $N \to \infty$ is given by

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as $m, n \to \infty$.

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Note that the estimate is sharp in the following sense: it's attained, and it needs *both* $n, m \to \infty$.

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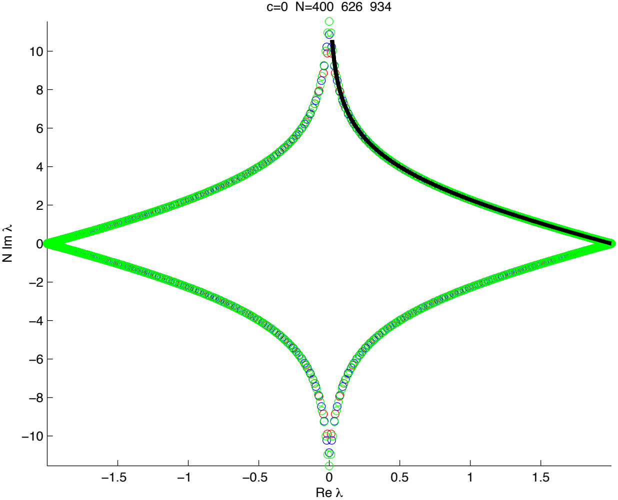

Asymptotics, $c = 0$, $n = m = N/2$

Theorem

Let $c = 0$, $n = m = N/2 \to \infty$. *The eigenvalues of* $\mathcal{P}_{n,n;0}$ *are all non-real, and satisfy*

$$\operatorname{Im}\lambda = \pm 1/N * Y(|\operatorname{Re}\lambda|) + o(N^{-1}),$$

where

$$Y(u) := \sqrt{4 - u^2}\,\log\cot(\pi/4 - \arccos(u/2)/2)$$

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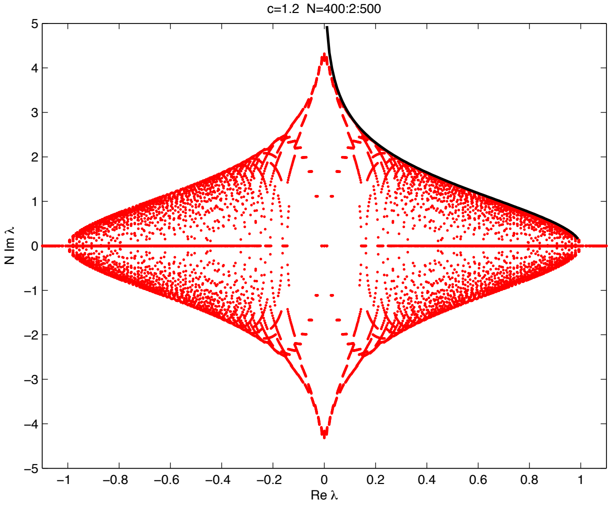

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where Y_c *is some explicitly described but complicated function.*

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Set $\lambda - c = z + 1/z$, $\lambda + c = w + 1/w$. Then for non-real eigenvalues

$$F_m(z)F_n(w) = -1,$$

where

$$F_m(z) = \frac{z^{n+1} - z^{-n-1}}{z^n - z^{-n}} = \frac{\sinh((n+1)\log z)}{\sinh(n \log z)}.$$

1d Dirac operator

Define a self-adjoint operator

$$\mathsf{T}_V = \begin{pmatrix} V+k & -\nabla \\ \nabla & V-k \end{pmatrix} = -i\sigma_2\nabla + k\sigma_3 + V,$$

where $\nabla = \frac{\mathrm{d}}{\mathrm{d}x}$, σ_2, σ_3 are Pauli matrices, k is the mass, and $V(x)$ is a potential.

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For a given potential V, we denote by Σ_V the spectrum of the linear operator pencil

$$\gamma \mapsto \mathsf{T}_0 + \gamma V = \begin{pmatrix} k & -\nabla \\ \nabla & -k \end{pmatrix} + \gamma \begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix}.$$

(The spectral parameter is denoted γ in this problem for historical reasons.)

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1d Dirac operator - history

Similar problems, as well as some other related questions, have been studied in a variety of situations in mathematical literature, e.g [Birman Solomyak 1977], [Klaus 1980], [Gesztesy et al. 1988], [Birman Laptev 1994], [Safronov 2001], [Schmidt 2010].

In physical literature, our problem appears in the study of electron waveguides in graphene (see [Hartmann Robinson Portnoi 2010], [Stone Downing Portnoi 2012] and many references there).

It was shown in [Hartmann Robinson Portnoi 2010] that for the potential $V_{\mathrm{HRP}}(x) = -1/\cosh(x)$ the solutions can be found explicitly in terms of special functions. Moreover, there exists an infinite sequence of coupling constants γ such that 0 is an eigenvalue of the operator $\mathrm{T}_{\gamma V_{\mathrm{HRP}}}$.

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Let $\mathbb{V}_1$ denote the class of real valued locally L^2 potentials which satisfy

$$\int_{\mathbb{R}} |V(x)| \,\mathrm{d}x < +\infty;$$

that is, we require V to be integrable. Equivalently, we can define $\mathbb{V}_1 = \mathbb{V}_0 \cap L^1$. The class $\mathbb{V}_1$ is sometimes denoted as $\ell^1(L^2)$.

General bounds

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Theorem

Suppose $V \in \mathbb{V}_1$. *Then*

$$\#\left(\Sigma_V \cap \{z \in \mathbb{C} : |z| \leq R\}\right) \leq C \, \|V\|_{L^1} R$$

for any $R \geq 0$, *where* C *is a universal constant (we can take* $C = 4\mathrm{e}/\pi$*).*

General bounds (contd.)

Restricting our attention to real points we have the following complementary lower bound

Theorem

Suppose $V \in \mathbb{V}_1$. *Then*

$$\#(\Sigma_V \cap [0, R]) \geq \frac{R}{\pi} \left| \int_{\mathbb{R}} V(x)\,\mathrm{d}x \right| + o(R)$$

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as $R \to \infty$, *while the same estimate holds for* $\#(\Sigma_V \cap [-R, 0])$ *(by symmetry). In particular,* $\Sigma_V \cap \mathbb{R}$ *contains infinitely many points if* $\int_{\mathbb{R}} V(x)\,\mathrm{d}x \neq 0$.

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Theorem also applies to potentials V satisfying the condition $V(a+x) = -V(a-x)$ for some $a \in \mathbb{R}$ and all $x \in \mathbb{R}$.

Discussion of the results

Our results give information about the asymptotics of the counting function $\#(\Sigma_V \cap [0, R])$ as $R \to \infty$. We've already seen two cases when the results give leading term asymptotic behaviour of

$$\frac{R}{\pi} \int_{\mathbb{R}} |V(x)| \, \mathrm{d}x \quad \text{and} \quad \frac{R}{\pi} \left| \int_{\mathbb{R}} V(x) \, \mathrm{d}x \right| \tag{2}$$

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respectively. (Though they coincide if V is sign-definite).

Discussion of the results (contd.)

The above results may lead to a hypothesis that, in fact, the lower bound always gives the leading order term in the asymptotics of the counting function of the spectrum. However, this is not the case; for general (variable-signed) potentials the precise asymptotic behaviour of $\#(\Sigma_V \cap [0, R])$ as $R \to \infty$ appears to depend on V in a rather subtle way. In particular, this behaviour appears to be sensitive to 'gaps' in the potential, namely intervals where $V \equiv 0$ appearing between components of $\mathrm{supp}(V)$.

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Surprise

We can construct potentials for which the actual asymptotic coefficient lies anywhere between the modulus of the integral of the potential and the L^1 norm, modulo multiplication by R/π.

Examples — general setup

We restrict our attention mostly to piecewise constant potentials with compact support; these allow the easiest analysis and already demonstrate the full range of effects. Consider points $a_0 < a_1 < \cdots < a_m$ which partition the real line into m finite intervals $I_j = (a_{j-1}, a_j)$, $j = 1, \ldots, m$, and two semi-infinite intervals $I_- = (-\infty, a_0)$ and $I_+ = (a_m, +\infty)$. Consider a potential

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$$V(x) = W\big(x; [a_0, \ldots, a_m]; \{v_1, \ldots, v_m\}\big) := \begin{cases} v_j, & x \in I_j,\ j = 1, \ldots, m, \\ 0, & x \in I_- \cup I_+, \end{cases} \tag{3}$$

with some given real constants v_j.

Examples — general setup (contd.)

On each interval, we can solve the equations explicitly in trigonometric functions; matching conditions lead to an explicit characteristic equation for eigenvalues: $\gamma \in \Sigma_V$ if and only if $D_V(\gamma) = 0$.

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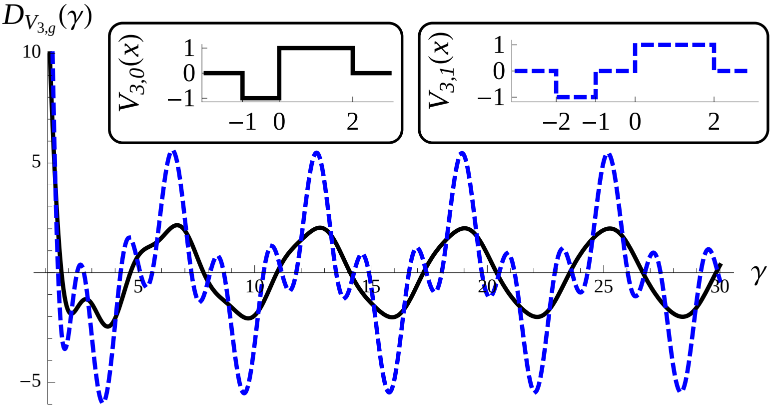
Thus, in each particular case our problem is reduced to constructing $D_V(\gamma)$ and finding its real or complex roots. We visualise the real roots of $D_V(\gamma)$ by simply plotting its graph for real arguments.

Example — One-gap non-zero-integral potentials

Consider the one-gap potentials $V_{3,g}(x) := W(x; [-g-1, -g, 0, 2]; \{-1, 0, 1\})$ parametrised by the gap length g. For each of these potentials, $\int_{\mathbb{R}} V_{3,g} = 1$ and $\|V_{3,g}\|_{L^1} = 3$. The graphs of $D_{V_{3,g}}(\gamma)$ for real γ and $g = 0$ or $g = 1$:



We can expect asymptotics of the form

$$\#(\Sigma_{V_{3,g}} \cap [0, R]) = C_g \frac{R}{\pi} + O(1),$$

as $R \to \infty$. For the no-gap potential $V_{3,0}$ one of our Theorems gives such an asymptotics with $C_0 = 1 = \int_{\mathbb{R}} V_{3,1}$. On the hand, $D_{V_{3,1}}(\gamma)$ has three times as many real roots as $D_{V_{3,0}}(\gamma)$ (for sufficiently large γ). This leads to a constant $C_1 = 3 = \|V_{3,0}\|_{L^1}$ in the asymptotics for the gap potential $V_{3,1}$.

Example — One-gap non-zero-integral potentials (contd.)

This example is just a partial case of a more complicated phenomenon Consider a general (not necessarily piecewise constant) one gap compact potential $V(x)$ such that $\mathrm{supp}(V) = I_1 \cup I_2$, where I_1 and I_2 are compact intervals separated by a gap of length $g > 0$, and assume additionally that $V(x)$ does not change sign on either I_j. If the signs of $V|_{I_1}$ and $V|_{I_2}$ coincide, then the asymptotic counting function involves $C = \|V\|_{L^1} = \left|\int_{\mathbb{R}} V\right|$. If, however, the signs of $V|_{I_1}$ and $V|_{I_2}$ are different, then the asymptotic behaviour is given by a complicated formula which depends not only upon the gap length g and the values of $\left|\int_{I_j} V\right|$ but also upon the rationality or irrationality of the ratio of these two integrals! The rigorous approach to this involves an intricate analysis based on the following version of a classical problem

Counting zeros

Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \cos(x) + a\cos(bx),$$

where a and b are real parameters satisfying $0 \le a < 1$ and $b > 0$. For any function $\phi : \mathbb{R} \to \mathbb{R}$ we also set $f_\phi = f + \phi$. We want to consider f_ϕ as a perturbation of $f = f_0$ for large x. To this end introduce the family of conditions

$$\phi \in C^k(\mathbb{R}), \quad \phi^{(k)}(x) = o(1) \text{ as } x \to \infty \tag{Ak}$$

where $k \in \mathbb{N}_0$ (we'll only need to consider $k = 0, 1, 2$).
Fix a perturbation ϕ. Introduce the counting function

$$\mathrm{N}_\phi(R) = \#\{x \in [0, R]), \mid f_\phi(x) = 0\} \in \mathbb{N} \cup \{0, \infty\}$$

We are interested in the asymptotic behaviour of $\mathrm{N}_\phi(R)$ as $R \to \infty$, and how this behaviour depends on the parameters a and b.

Counting zeros — small ab

Proposition

Suppose $ab < 1$ and ϕ satisfies (A0), (A1). Then

$$\mathrm{N}_\phi(R) = \frac{1}{\pi} R + O(1) \quad \text{as } R \to \infty.$$

Remark

When $ab < 1$ we get the same asymptotic behaviour for $\mathrm{N}_\phi(R)$ as in the case $a = 0$ (that is, when $f = \cos$).

Counting zeros — large ab, irrational case

When $ab > 1$ we can define $\alpha, \beta \in (0, \pi/2)$ by

$$\alpha = \arcsin \frac{\sqrt{a^2 b^2 - 1}}{\sqrt{b^2 - 1}} \quad \text{and} \quad \beta = \arcsin \frac{\sqrt{1 - a^2}}{a\sqrt{b^2 - 1}}. \tag{4}$$

Also set $u = \frac{2}{\pi}(b\alpha + \beta)$. If we fix $b > 1$ and vary a from $1/b$ to 1 it is easy to check that α increases from 0 to $\pi/2$ and β decreases from $\pi/2$ to 0; it follows that u varies from 1 to b.

Proposition

Suppose $ab > 1$, b is irrational and ϕ satisfies (A0), (A1), (A2). Then

$$\lim_{R\to\infty} \frac{\mathrm{N}_\phi(R)}{R} = \frac{1}{\pi} u.$$

Counting zeros — large ab, rational case

Proposition

Suppose $ab > 1$, b is rational and ϕ satisfies (A0), (A1). Write $b = p/q$ where $p, q \in \mathbb{N}$ are coprime. If p and q are odd set $P = p$ and $Q = q$; if p and q have opposite parity set $P = 2p$ and $Q = 2q$. If $P + Qu \notin 4\mathbb{Z}$ then

$$\lim_{R\to\infty} \frac{\mathrm{N}_\phi(R)}{R} = \frac{1}{\pi}\left(\frac{4}{Q}\left\lfloor \frac{1}{4}(P+Qu)\right\rfloor - \frac{P}{Q} + \frac{2}{Q}\right). \tag{5}$$

We are using $\lfloor x \rfloor$ to denote the largest integer which does not exceed x.

Counting zeros — large ab, rational case (contd.)

Remark

From (5) and the bounds $x - 1 \leq \lfloor x \rfloor \leq x$ we get

$$\frac{1}{\pi} u - \frac{2}{Q\pi} \leq \lim_{R \to \infty} \frac{\mathrm{N}_\phi(R)}{R} \leq \frac{1}{\pi} u + \frac{2}{Q\pi}.$$

Using the size of Q as a measure of 'how irrational' b is it follows that the result for irrational b can be viewed as a limit of the rational case.