On Inaudible Properties of Broken Drums

Isospectrality with Mixed Dirichlet-Neumann Boundary Conditions

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Spectral Theory of Laplace and Schrödinger Operators
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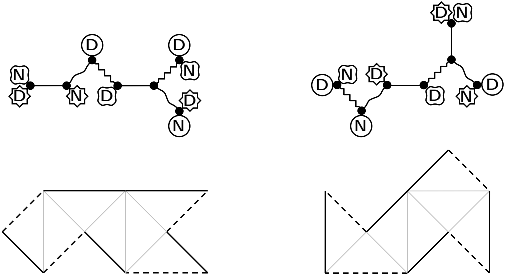

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- M a compact flat manifold with piecewise smooth boundary ∂M
- $\partial M = \overline{\partial_D M \cup \partial_N M}$ for some open, smooth, disjoint $\partial_D M, \partial_N M$

Zaremba Problem: $\varphi \in C^0(M) \cap C^\infty(M^\circ \cup \partial_D M \cup \partial_N M)$

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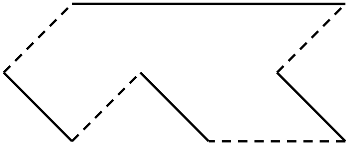

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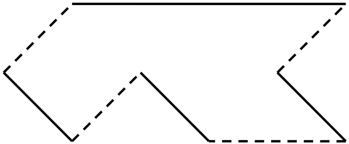

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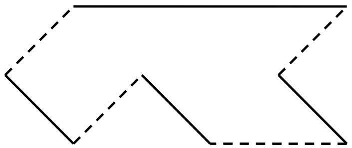

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Global Assumption

These mixed boundary conditions give rise to an extension of $\Delta_{|C_0^\infty(M^\circ)}$ that is self-adjoint and has discrete spectrum

$$0 \le \lambda_0 \le \lambda_1 \le \dots$$

In particular, $L^2(M)$ is the Hilbert direct sum of its eigenspaces.

Inverse Spectral Geometry

Spectral invariants (obtained form heat kernel expansions):

- dim (M)
- vol (M)
- vol $(\partial_D M)$ – vol $(\partial_N M)$
- Number of components of M if $\partial M = \overline{\partial_N M}$

Levitin, Parnovski, Polterovich (2006)

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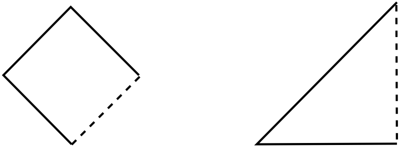
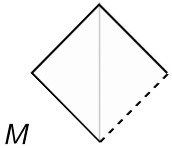


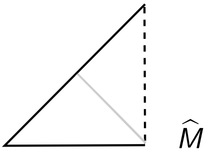
The Cut and Paste Proof

Buser (1986)

Transplantation method: Cut eigenfunctions on M into pieces φ_j and superpose these restrictions linearly on the blocks of $\widehat{M}$

$$\widehat{\varphi}_i = \sum_j T_{ij}\,\varphi_j$$





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- $\widehat{\varphi}$ satisfies boundary conditions (solves Zaremba problem on $\widehat{M}$)
- $T = (T_{ij})$ is invertible and the inverse transplantation T^{-1} equally maps eigenfunctions of $\widehat{\Delta}$ to eigenfunctions of Δ

Then

- T and T^{-1} map eigenspaces of Δ and $\widehat{\Delta}$ into each other, hence
- $\text{spec}(\Delta) = \text{spec}(\widehat{\Delta})$, that is, M and $\widehat{M}$ are isospectral
- T is said to be intertwining ($T \circ \Delta = \widehat{\Delta} \circ T$)

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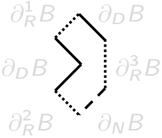
Building Blocks, Reflecting Faces and Tiled Manifolds

- **Building block**: Compact flat manifold B with piecewise smooth boundary $\partial B = \left(\bigcup_{i=1}^{C} \partial_R^i B\right) \cup \partial_D B \cup \partial_N B$ having open smooth
- **Reflecting faces**: $\partial_R^i B \subseteq \partial B$ each of which has a neighbourhood in B isometric to an open subset of closed Euclidean upper half space
- **Tiled manifold**: M is obtained by gluing copies B_i of B along pairs $((\partial_R^{c_k} B_{i_k}, \partial_R^{c_k} B_{j_k}))_k$ such that $M^\circ = \bigcup_i B_i^\circ \cup \bigcup_k \partial_R^{c_k} B_{i_k}$
- **Boundary conditions**: either originate from B or are selectable



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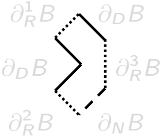

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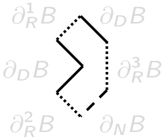

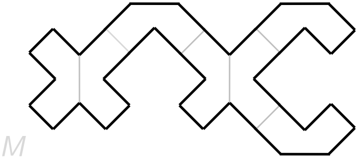

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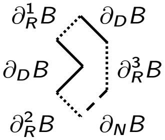

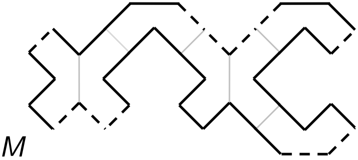

Extending Eigenfunctions

Let

- B be a building block with reflecting face $\partial_R B$
- $\varphi \in C^\infty(B^\circ \cup \partial_R B)$ be an eigenfunction of Δ on B°
- φ satisfy Neumann (Dirichlet) boundary conditions on $\partial_R B$

Reflection Principle

φ can be continued across $\partial_R B$ by itself (by $-\varphi$) to a smooth function.

Unique Continuation Theorem

There is at most one continuation across $\partial_R B$ to a smooth eigenfunction.

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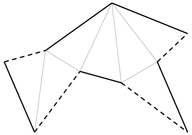
Buser (1988), Okada and Shudo (2001), Herbrich (2009)

Tiled manifolds can be encoded by edge-coloured loop-signed graphs

Building blocks	$\Longleftrightarrow$	Vertices
Glued reflecting faces	$\Longleftrightarrow$	Links
Unglued reflecting faces	$\Longleftrightarrow$	Loops
Indices of reflecting faces	$\Longleftrightarrow$	Edge colours
Boundary conditions	$\Longleftrightarrow$	Loop signs



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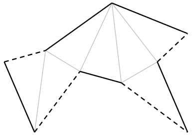

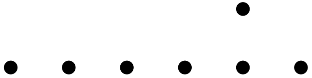

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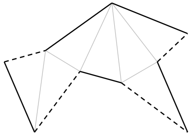

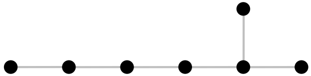

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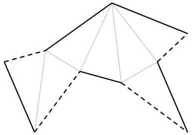

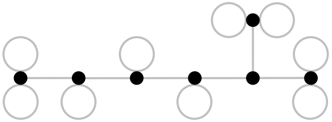

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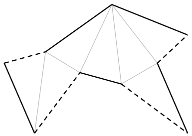

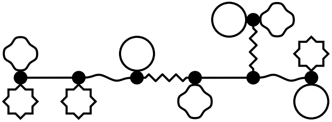

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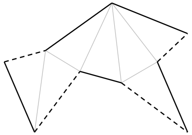

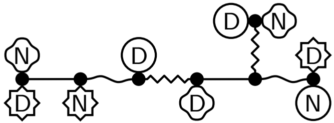

Transplantations in Terms of Graphs

Loop-signed graphs are encoded by adjacency matrices A^c:

Non-vanishing off-diagonal entries (connectivity)

$$A^c_{ij} = 1 \text{ if vertices } i \text{ and } j \text{ are joined by a } c\text{-coloured edge}$$

Non-vanishing diagonal entries (boundary conditions)

$$A^c_{ii} = \begin{cases} 1 & \text{if vertex } i \text{ has a } c\text{-coloured loop with sign } N \\ -1 & \text{if vertex } i \text{ has a } c\text{-coloured loop with sign } D \end{cases}$$

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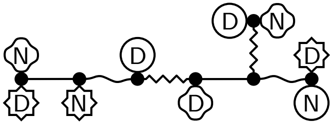
$$A^{\text{straight}} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$



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Non-vanishing diagonal entries (boundary conditions)

$$A_{ii}^{\mathbf{c}} = \begin{cases} 1 & \text{if vertex } i \text{ has a } \boldsymbol{c}\text{-coloured loop with sign } N \\ -1 & \text{if vertex } i \text{ has a } \boldsymbol{c}\text{-coloured loop with sign } D \end{cases}$$

Reflection Principle and Unique Continuation Theorem

If φ is a solution of the Zaremba problem given by $(A^{\boldsymbol{c}})_{\boldsymbol{c}=1}^{C}$, then $\sum_l A_{kl}^{\boldsymbol{c}} \varphi_l$ is the smooth extension of its restriction φ_k across the $\boldsymbol{c}$-face.

Transplantations in Terms of Graphs

Herbrich (2009)

Let $(A^{c})_{c=1}^{C}$ and $(\widehat{A}^{c})_{c=1}^{C}$ describe Zaremba problems on tiled manifolds.
Then, $T = (T_{ij})$ is intertwining if and only if

$$\widehat{A}^{c} = T\,A^{c}\,T^{-1} \qquad \text{for } c = 1, \ldots, C.$$

For Neumann graphs, $\mathrm{Tr}(A^{c_1}\,A^{c_2}\,\cdots\,A^{c_l})$ equals the number of closed paths on the graph with colour sequence $c_1 c_2 \ldots c_l$.

Okada and Shudo (2001), Herbrich (2009)

Graphs are transplantable if and only if for all finite sequences $c_1 c_2 \ldots c_l$

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Sunada's Method

Gassmann Triple: Finite group G with subgroups H and $\widehat{H}$ satisfying

$$|\,[g] \cap H| = |\,[g] \cap \widehat{H}| \text{ for all } g \in G.$$

$(G, H, \widehat{H})$ is Gassmann if and only if $\mathrm{Ind}_H^G(\mathbf{1}_H) \simeq \mathrm{Ind}_{\widehat{H}}^G(\mathbf{1}_{\widehat{H}})$.

Sunada (1985)

M a closed Riemannian manifold

- G a finite group acting freely on M by isometries
- H and $\widehat{H}$ subgroups of G such that $(G, H, \widehat{H})$ is Gassmann

Then, M/H and $M/\widehat{H}$ are isospectral.

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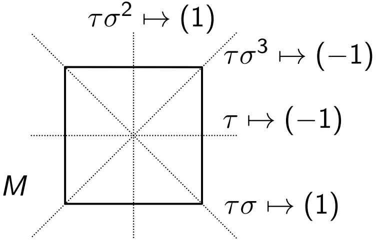
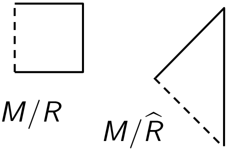
Isospectrality and Induced Representations

Band, Parzanchevski, Ben-Shach (2009)

If $\mathrm{Ind}_H^G(R) \simeq \mathrm{Ind}_{\widehat{H}}^G(\widehat{R})$, then M/R and $M/\widehat{R}$ are isospectral.





$$G = \mathrm{Isom}(M) = D_4 = \{e, \sigma, \sigma^2, \sigma^3, \tau, \tau\sigma, \tau\sigma^2, \tau\sigma^3\}$$

$$H = \{e, \tau, \tau\sigma^2, \sigma^2\} \quad R: \{e \mapsto 1, \tau \mapsto -1, \tau\sigma^2 \mapsto 1, \sigma^2 \mapsto -1\}$$

$$\widehat{H} = \{e, \tau\sigma, \tau\sigma^3, \sigma^2\} \quad \widehat{R}: \{e \mapsto 1, \tau\sigma \mapsto 1, \tau\sigma^3 \mapsto -1, \sigma^2 \mapsto -1\}$$

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Herbrich (2012)

Each pair of transplantable loop-signed graphs gives rise to a triple

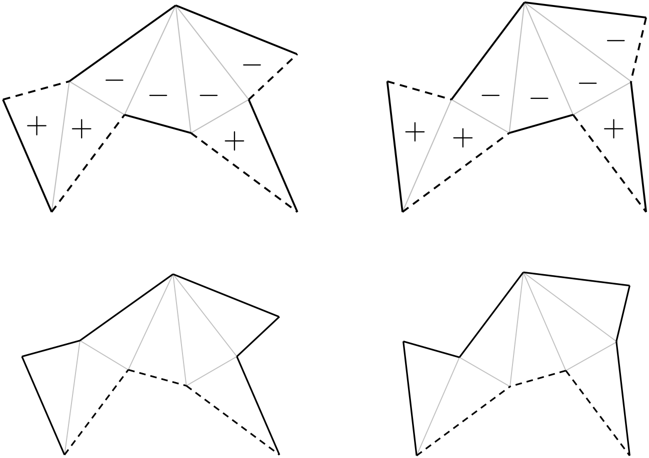
$$(G, ((H_i, R_i))_i, ((\widehat{H}_j, \widehat{R}_j))_j)$$

such that

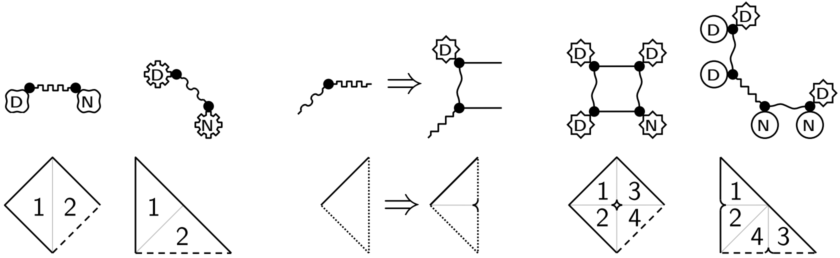
$$\bigoplus_i \mathrm{Ind}_{H_i}^G(R_i) \simeq \bigoplus_j \mathrm{Ind}_{\widehat{H}_j}^G(\widehat{R}_j),$$

and the pair can be recovered from the triple up to isomorphism.

Dualisation



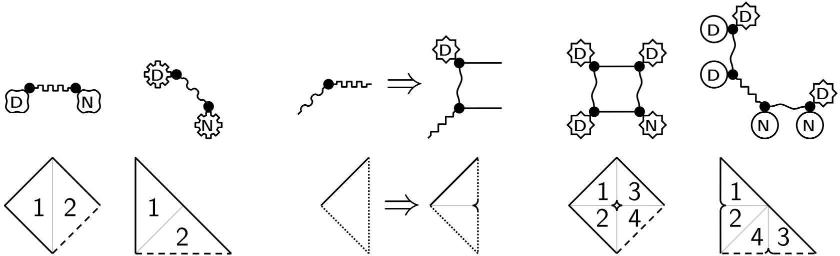
Substitution Method



Substitutions yield wreath products with imprimitive actions.

There are infinitely many transplantable pairs.

Substitution Method



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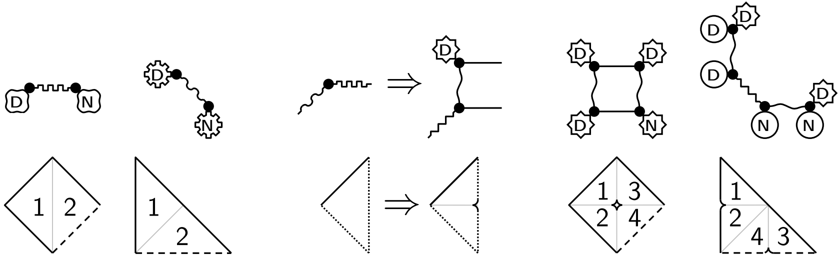

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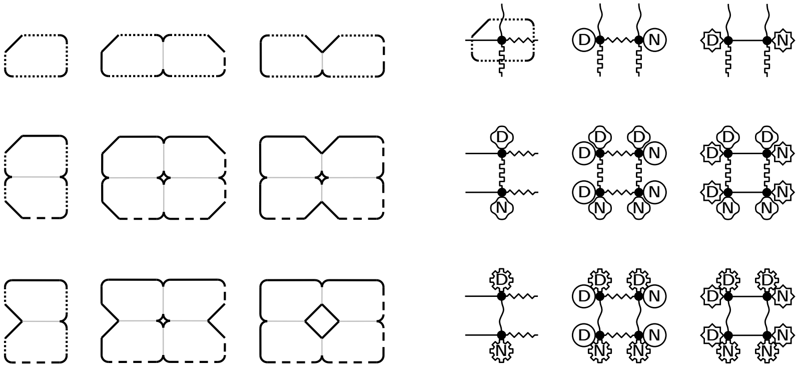
Pairwise Transplantable Tuples



Levitin, Parnovski, Polterovich (2006)

The Algorithm

1. **Graph generation**: Ordered walk on the graph
2. **Graph hashing**: Trace condition (2009) or

P. Doyle (2010)

Transplantability is

"a non-commutative version of strong isospectrality".

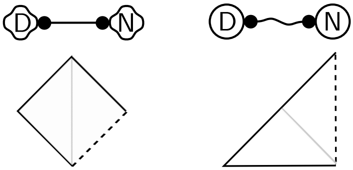
$$P(z_1, z_2, \ldots, z_C) = \det\left(\sum_{\boldsymbol{c}=1}^{C} z_{\boldsymbol{c}} A^{\boldsymbol{c}}\right)$$

$$P(Z_1, Z_2, \ldots, Z_C) = \mathrm{Tr}\left(\left(\sum_{\boldsymbol{c}=1}^{C} Z_{\boldsymbol{c}} \otimes A^{\boldsymbol{c}}\right)^k\right)$$

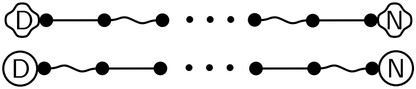
$$= \sum_{1 \le \boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_k \le C} \mathrm{Tr}\left(\prod_{j=1}^{k} A^{\boldsymbol{c}_j}\right) \mathrm{Tr}\left(\prod_{j=1}^{k} Z_{\boldsymbol{c}_j}\right)$$

3. **Graph sorting**: Merge sort

Pairs with 2 Edge Colours



Levitin, Parnovski, Polterovich (2006)



In general

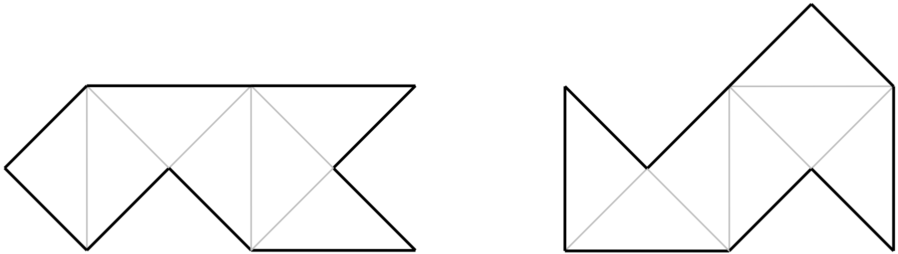
Pairs with 3 Edge Colours

Number of	Loop-signed		Transplantable		Transplantable	
Vertices	Graphs	(Treelike)	Pairs	(Treelike)	Classes	(Treelike)
2	40	(30)	9	(6)	3	(2)
3	128	(96)	0	(0)	0	(0)
4	737	(472)	118	(64)	28	(18)
5	3 848	(2 304)	0	(0)	0	(0)
6	24 360	(12 792)	957	(294)	176	(56)
7	156 480	(73 216)	112	(112)	32	(32)
8	1 076 984	(439 968)	13 349	(2 112)	2 343	(375)
9	7 625 040	(2 715 648)	0	(0)	0	(0)
10	55 931 952	(17 203 136)	?	?	?	?
11	420 522 592	(111 132 672)	?	?	?	?
12	3 238 019 281	(730 325 760)	?	?	?	?
13	25 434 892 136	(4 868 669 440)	?	?	?	?

Pairs with 3 Edge Colours

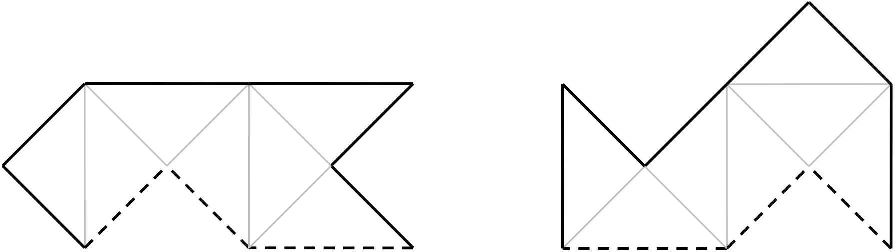
Number of Vertices	Edge-coloured Graphs	Edge-coloured Trees	Dirichlet Pairs	Dirichlet Classes	Neumann Pairs	Neumann Classes	Treelike Pairs	Treelike Classes
7	1 407	143	7	3	7	3	7	3
8	6 877	450	64	16	28	8	0	0
9	28 665	1 326	0	0	0	0	0	0
10	142 449	4 262	0	0	0	0	0	0
11	681 467	13 566	34	9	70	19	0	0
12	3 535 172	44 772	2 362	440	42	10	0	0
13	18 329 101	148 580	26	9	26	9	26	9
14	99 531 092	502 101	345	77	798	163	42	7
15	546 618 491	1 710 855	51	13	159	33	15	4
16	3 098 961 399	5 895 090	?	?	?	?	?	?
17	17 827 256 505	20 470 230	?	?	?	?	?	?

Broken Gordon-Webb-Wolpert Drums



Gordon, Webb, Wolpert (1992)

Broken Gordon-Webb-Wolpert Drums



Band, Parzanchevski, Ben-Shach (2009)

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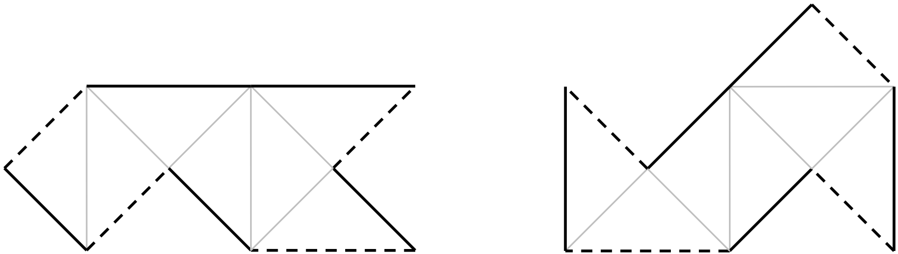
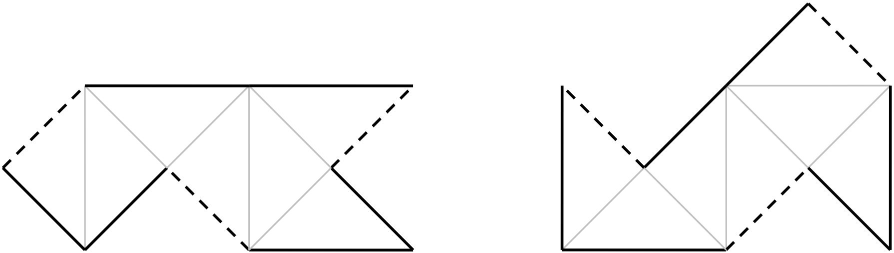


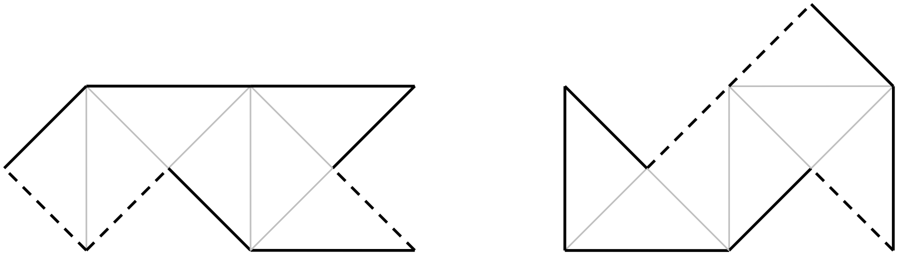
Conjectured by Driscoll, Gottlieb (2003)

Broken Gordon-Webb-Wolpert Drums



New isospectral pair

Broken Gordon-Webb-Wolpert Drums



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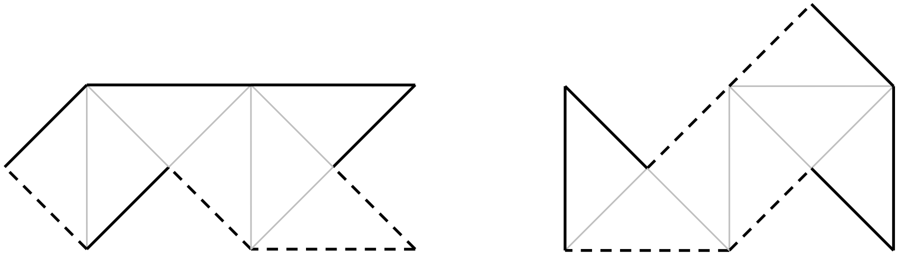
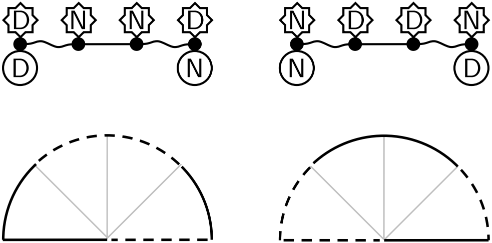


New isospectral pair

Self-Dual Pairs

One cannot “hear” which parts are broken!



Jakobson et al. (2006)

Self-Dual Pairs

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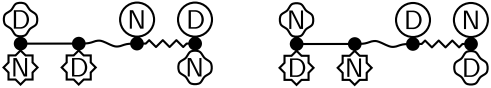

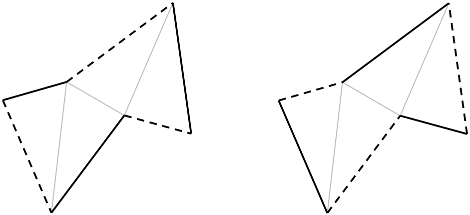

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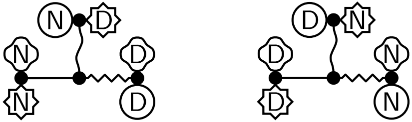
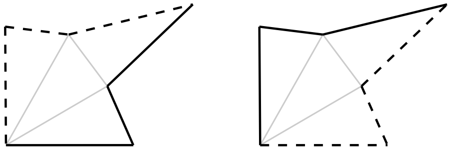
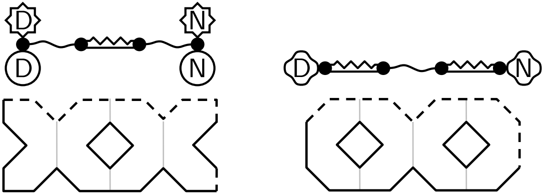


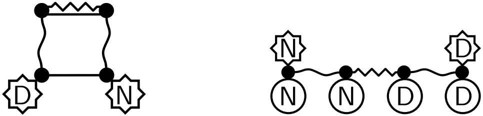


Fundamental Group

One cannot “hear” the fundamental group of broken drums!

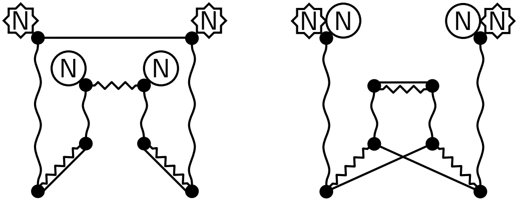


Levitin, Parnovski, Polterovich (2006)



Orientability

One cannot “hear” whether a broken drum is orientable!



Bérard, Webb (1995)

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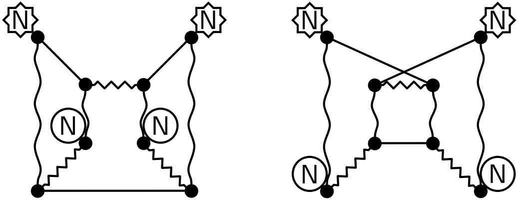

New pair obtained by braiding

Orientability

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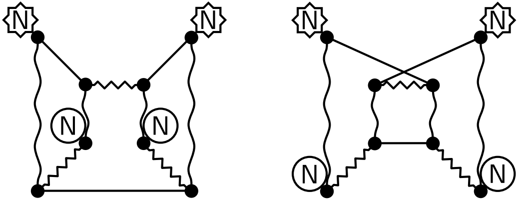

P. Doyle (2010)

There is no such pair of connected Dirichlet graphs.

Orientability

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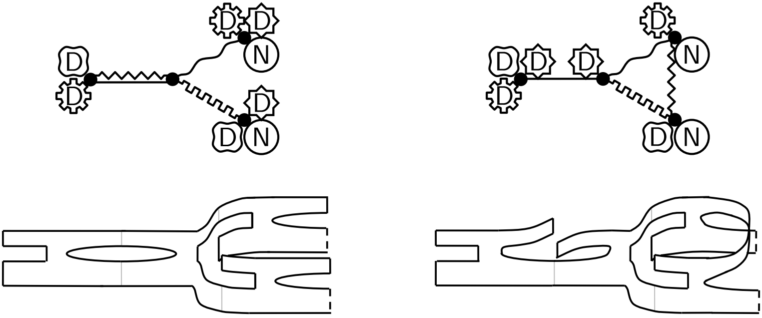
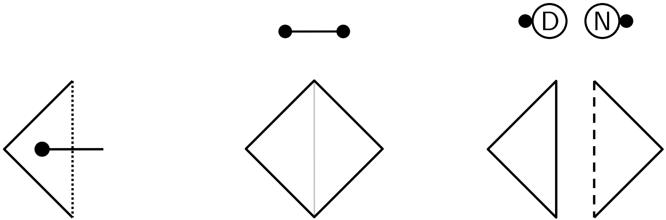


Connectedness

One cannot "hear" whether a drum is connected!



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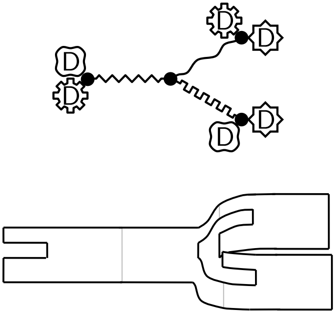
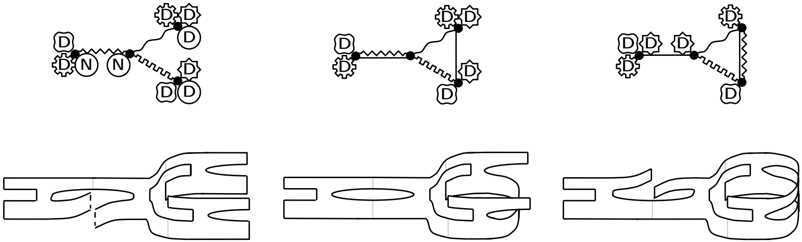


Isotropy Order

One cannot "hear" whether a drum is broken!



An orbifold can be Dirichlet isospectral to a manifold!

Isotropy Order

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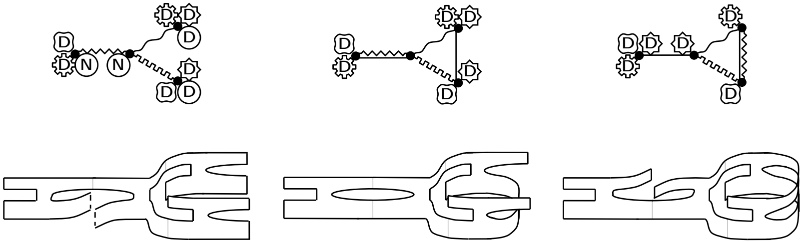

An orbifold can be Dirichlet isospectral to a manifold!

Thank you for your attention!

P. Herbrich, *On Inaudible Properties of Broken Drums - Isospectral Domains with Mixed Boundary Conditions*, preprint arXiv:1111.6789v2

Extending Eigenfunctions

Proof of Reflection Principle.

Using local coordinates $(x_1, x_2, \ldots, x_d)$, it suffices to prove:
If

- $\varphi \in C^\infty\left((-l,l)^{d-1} \times (-l,0]\right)$ for some $l > 0$,
- $\Delta\varphi = \lambda\varphi$ on $(-l,l)^{d-1} \times (-l,0)$, and
- $\frac{\partial\varphi}{\partial x_d}|_{x_d=0} \equiv 0 \qquad (\varphi|_{x_d=0} \equiv 0)$,

then φ can be extended to a smooth function on $(-l,l)^d$ by setting

$$\varphi(x_1,\ldots,x_{d-1},x_d) = \pm\varphi(x_1,\ldots,x_{d-1},-x_d) \qquad \text{for } x_d > 0.$$

This follows from elliptic regularity theory since $\varphi \in C^1((-l,l)^d)$ and therefore it is a weak solution of $(\Delta - \lambda)\varphi = 0$ on $(-l,l)^d$. □

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If

- $\varphi \in C^\infty\left((-l,l)^{d-1} \times (-l,0]\right)$ for some $l > 0$,
- $\Delta\varphi = \lambda\varphi$ on $(-l,l)^{d-1} \times (-l,0)$, and
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then φ can be extended to a smooth function on $(-l,l)^d$ by setting

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This follows from elliptic regularity theory since $\varphi \in C^1((-l,l)^d)$ and therefore it is a weak solution of $(\Delta - \lambda)\varphi = 0$ on $(-l,l)^d$. □

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Transplantations in Terms of Graphs

Proof of Transplantation Theorem.

- If $\widehat{A}^{\boldsymbol{c}}\, T = T\, A^{\boldsymbol{c}}$ for all $\boldsymbol{c}$, and if φ solves the Zaremba problem given by $(A^{\boldsymbol{c}})_{\boldsymbol{c}=1}^{C}$, then $\widehat{\varphi}$ with $\widehat{\varphi}_i = \sum_k T_{ik}\, \varphi_k$ solves the Zaremba problem given by $(\widehat{A}^{\boldsymbol{c}})_{\boldsymbol{c}=1}^{C}$ since $\widehat{A}_{ij}^{\boldsymbol{c}} = \pm 1$ implies

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- If T is intertwining, then for all solutions φ of the Zaremba problem given by $(A^{\boldsymbol{c}})_{\boldsymbol{c}=1}^{C}$,

$$T\, A^{\boldsymbol{c}}\, \varphi = \widehat{A}^{\boldsymbol{c}}\, \widehat{\varphi} = \widehat{A}^{\boldsymbol{c}}\, T\, \varphi, \text{ hence } T\, A^{\boldsymbol{c}} - \widehat{A}^{\boldsymbol{c}}\, T = 0.$$

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Transplantations in Terms of Graphs

Proof of Trace Theorem.

$G = \langle A^1, A^2, \ldots, A^C \rangle$ and $\widehat{G} = \langle \widehat{A}^1, \widehat{A}^2, \ldots, \widehat{A}^C \rangle$ are finite since they act faithfully on

$$\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_V, -\boldsymbol{e}_1, -\boldsymbol{e}_2, \ldots, -\boldsymbol{e}_V\}.$$

Define $\Phi : F^C \twoheadrightarrow G$ via $\Phi(c_1^{\pm 1} \ldots c_l^{\pm 1}) = A^{c_l} \cdots A^{c_1}$, similarly $\widehat{\Phi}$.

Since $\ker(\Phi) = \{w \in F^C \mid \mathrm{Tr}(\Phi(w)) = V\} = \ker(\widehat{\Phi})$, we have $G \simeq F^C / \ker(\Phi) \simeq \widehat{G}$ with isomorphism $\mathcal{I}(\Phi(.)) = \widehat{\Phi}(.)$.

The representations $id : G \to GL(\mathbb{C}^V)$ and $\widehat{id} \circ \mathcal{I} : G \to GL(\mathbb{C}^V)$ have equal characters, so there exists T with $T\,\Phi(.) = \widehat{\Phi}(.)\,T$.

There is $t \in \mathbb{R}$ such that $\mathrm{Re}(T) + t\,\mathrm{Im}(T)$ is invertible. □

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Isospectrality and Induced Representations

Obtaining Group Data.

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Choose one vertex v_i, resp. $\widehat{v}_j$, in each connected component.

Each A^c and $\widehat{A}^c$ acts on $\{\{e_1, -e_1\}, \{e_2, -e_2\}, \dots, \{e_V, -e_V\}\}$.

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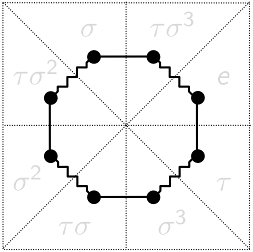
Cayley and Schreier Coset Graphs

G a finite group generated by involutions $(\gamma^{\boldsymbol{c}})_{\boldsymbol{c}=1}^{C}$

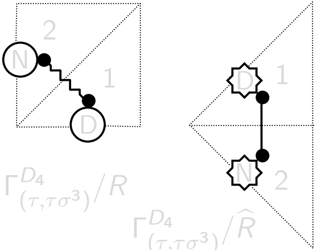
H a subgroup with real linear representation R

Schreier coset graph $\Gamma^{G}_{(\gamma^c)_{c=1}^{C}}/R$ with vertices $\{Hg \mid g \in G\}$

- $Hg \overset{\boldsymbol{c}}{\longleftrightarrow} Hg' \Leftrightarrow Hg\,\gamma^{\boldsymbol{c}} = Hg'$
- If $Hg\,\gamma^{\boldsymbol{c}} = Hg$, then Hg has a $\boldsymbol{c}$-coloured loop ($R(g\,\gamma^{\boldsymbol{c}}\,g^{-1}) = \pm 1$)



Cayley graph $\Gamma^{D_4}_{(\tau,\tau\sigma^3)}$



$\Gamma^{D_4}_{(\tau,\tau\sigma^3)}/R$

$\Gamma^{D_4}_{(\tau,\tau\sigma^3)}/\widehat{R}$

Cayley and Schreier Coset Graphs

G a finite group generated by involutions $(\gamma^{c})_{c=1}^{C}$
H a subgroup with real linear representation R

Schreier coset graph $\Gamma^{G}_{(\gamma^c)_{c=1}^{C}}/R$ with vertices $\{Hg \mid g \in G\}$

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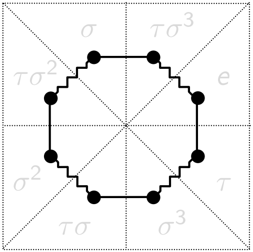
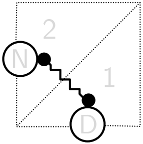
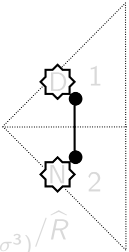
- If $Hg\,\gamma^{c} = Hg$, then Hg has a c-coloured loop $(R(g\,\gamma^{c}\,g^{-1}) = \pm 1)$



Cayley graph $\Gamma^{D_4}_{(\tau,\tau\sigma^3)}$



$\Gamma^{D_4}_{(\tau,\tau\sigma^3)}/R$



$\Gamma^{D_4}_{(\tau,\tau\sigma^3)}/\widehat{R}$

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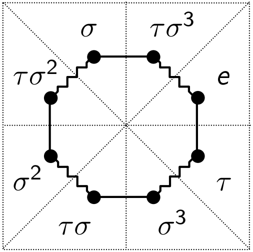
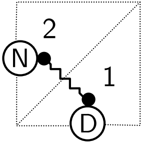
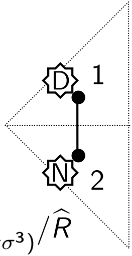


Cayley graph $\Gamma^{D_4}_{(\tau,\tau\sigma^3)}$



$\Gamma^{D_4}_{(\tau,\tau\sigma^3)}/R$



$\Gamma^{D_4}_{(\tau,\tau\sigma^3)}/\widehat{R}$