Global stability of the normal state of superconductors in the presence of a strong electric current
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Time Dependent Ginzburg-Landau equation.

Consider a superconductor placed at a temperature lower than the critical one. It is well-understood from numerous experimental observations, that a sufficiently strong current, applied through the sample, will force the superconductor to arrive at the normal state. To explain this phenomenon mathematically, we use the time-dependent Ginzburg-Landau model which is defined by the following system of equations, and will be referred to as (TDGL1) (Time Dependent Ginzburg-Landau equation).

(TDGL1)

$$\frac{\partial \psi}{\partial t} + i\phi\psi = (\nabla - iA)^2 \psi + \psi \left(1 - |\psi|^2\right) , \qquad \text{in } \mathbb{R}_+ \times \Omega \,, \tag{1a}$$

$$\kappa^2 \operatorname{curl}{}^2 A + \sigma \left(\frac{\partial A}{\partial t} + \nabla \phi \right) = \operatorname{Im} (\bar{\psi} \cdot (\nabla - iA)\psi) \,, \qquad \text{in } \mathbb{R}_+ \times \Omega \,, \tag{1b}$$

$$\psi = 0 \,, \qquad \text{on } \mathbb{R}_+ \times \partial\Omega_c \,, \tag{1c}$$

$$(\nabla - iA)\psi \cdot \nu = 0 \,, \qquad \text{on } \mathbb{R}_+ \times \partial\Omega_i \,, \tag{1d}$$

$$\sigma \left(\frac{\partial A}{\partial t} + \nabla \phi \right) \cdot \nu = J \,, \qquad \text{on } \mathbb{R}_+ \times \partial\Omega_c \,, \tag{1e}$$

$$\sigma \left(\frac{\partial A}{\partial t} + \nabla \phi \right) \cdot \nu = 0 \,, \qquad \text{on } \mathbb{R}_+ \times \partial\Omega_i \,. \tag{1f}$$

$$\frac{1}{|\partial\Omega|}\int_{\partial\Omega} \operatorname{curl} A(t,x)\, ds = h_{ex}\,, \text{ on } \mathbb{R}_+\,, \quad (1g)$$

$$\psi(0,x) = \psi_0(x)\,, \qquad \text{in } \Omega\,, \quad (1h)$$

$$A(0,x) = A_0(x)\,, \qquad \text{in } \Omega\,, \quad (1i)\,.$$

In the above ψ denotes the order parameter, A is the magnetic potential, ϕ is the electric potential, κ denotes the Ginzburg-Landau parameter, which is a material property, and the normal conductivity of the sample is denoted by σ. ds denotes the induced measure on $\partial\Omega$. The domain $\Omega \subset\subset \mathbb{R}^2$, occupied by the superconducting sample, has a smooth interface, denoted by $\partial\Omega_c$, with a conducting metal which is at the normal state.

We require that ψ would vanish on $\partial\Omega_c$ in (1c), and allow for a smooth current

$$J = jJ_r\,,$$

satisfying

$$(J) \quad \int_{\partial\Omega_c} J_r\, ds = 0\,, \tag{3}$$

and

$$(2) \quad \text{the sign of } J_r \text{ is constant on each connected component of } \partial\Omega_c\,. \tag{4}$$

We also require:

$$\text{Both } \partial\Omega_c \text{ and } \partial\Omega_i \text{ have two components.} \tag{5}$$

Figure 1 presents a typical sample.

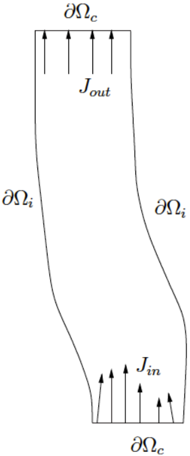

FIGURE 1. Typical superconducting sample. The arrows denote the direction of the current flow (J_{in} for the inlet, and J_{out} for the outlet).

We assume, for the initial conditions (1h,i), that

$$\psi_0 \in H^1(\Omega,\mathbb{C}) \text{ and } A_0 \in H^1(\Omega,\mathbb{R}^2)\,, \tag{6}$$

and:

$$\|\psi_0\|_\infty \leq 1\,. \tag{7}$$

We consider Coulomb gauge solutions of (1):

$$\operatorname{div} A = 0 \ \text{ in } \Omega,\ A\cdot\nu = 0 \text{ on } \partial\Omega\,. \tag{8}$$

Stationary normal solutions.

If we assume time independence and a solution of (TDGL1) $(0, A_n, \phi_n)$, we get in the Coulomb gauge for the magnetic and electric normal potentials A_n and ϕ_n. These equations are obtained by setting $\psi \equiv 0$ in (1b), yielding

$$\begin{cases} -c \ \text{curl}\ ^2 A_n + \nabla \phi_n = 0 & \text{in } \Omega\,, \\ -\sigma \frac{\partial \phi_n}{\partial \nu} = j\, J_r & \text{on } \partial\Omega\,, \\ \frac{1}{|\partial\Omega|} \int_{\partial\Omega} \ \text{curl}\ A_n\, ds = h_{ex}\,, & \end{cases}$$

in which $c = \kappa^2/\sigma$.

One possible mechanism which contributes to the breakdown of superconductivity by a strong current is the magnetic field induced by the current. In the absence of electric current, it was proved by Giorgi-Phillips that, when a sufficiently strong magnetic field is applied on the sample's boundary (or when h_{ex} is sufficiently large), the normal state, for which $\psi \equiv 0$, becomes the unique solution for the steady-state version of (1) (cf. also Fournais-Helffer and the references therein).
For the time-dependent Ginzburg-Landau equations it was proved in Feireisl-Takac that every solution must reach an equilibrium in the long-time limit. When combined with the results in Giorgii-Phillips, it follows that when the applied magnetic field is sufficiently large the normal state becomes globally stable.

Much less is known in the presence of electric currents. Moreover, the magnetic field is not the only mechanism which forces the sample into the normal state when the electric current is sufficiently large.
Prove global stability of the normal state, as a solution of (1), for sufficiently large currents. Determine the right notions of critical fields or of critical curves

$$f(h_{ex}, j, \kappa, \sigma) = 0\,,$$

determining the stability or not of the solutions.

A non self-adjoint operator.

The linearization of the problem at the normal stationary state leads to the analysis: Let

$$\mathcal{L}_h = -\nabla^2_{hA_n} + i\, h\phi_n \,,$$

be defined over the domain

$$D(\mathcal{L}_h) = \{u \in H^2(\Omega) \mid u|_{\partial\Omega_c} = 0 \,;\ \nabla u \cdot \nu|_{\partial\Omega_i} = 0\,\}\,.$$

The goal is to prove that a proper bound on the resolvent of $\mathcal{L}_h$.

The general question is to prove that if the current is strong enough, the magnetic field induced by this current forces the stability, that is the convergence as $t \to +\infty$ to the normal stationary state. Let

$$\mu(h) = \inf_{\substack{u \in H^1(\Omega,\mathbb{C}) \\ u|_{\partial\Omega_c}=0\ ;\ \|u\|_2=1}} \|\nabla_{hA_n} u\|_2^2 \,.$$

This is simply the ground state energy of the magnetic Laplacian (selfadjoint part of $\mathcal{L}_h$).

In some asymptotic regime, the following model plays a role and one open question is:
Show that the spectrum of

$$(D_y - \frac{x^2}{2})^2 + D_x^2 + icy$$

in $\mathbb{R}^2_+$ (Dirichlet) is non empty when $c \neq 0$.
Known results (Almog-Helffer-Pan) are for c small and c large.

References : Fournais-Helffer (book), Almog, Almog-Helffer-Pan, Almog-Helffer.

Bibliography

Y. ALMOG, *The stability of the normal state of superconductors in the presence of electric currents*, SIAM Journal on Mathematical Analysis, 40 (2008), pp. 824–850.

Y. ALMOG, B. HELFFER, AND X. PAN, *Superconductivity near the normal state in a half-plane under the action of a perpendicular electric current and an induced magnetic field*. Available at http://www.math.lsu.edu/∼almog/public_ps/halfplane-11c.pdf.

———, *Superconductivity near the normal state in a half-plane under the action of a perpendicular electric current and an induced magnetic field II : The large conductivity limit*. Available at http://www.math.lsu.edu/ almog/public_ps/largecon7d.pdf.

———, *Superconductivity near the normal state under the action of electric currents and induced magnetic fields in* $\mathbb{R}^2$, Comm. Math. Phys., 300 (2010), pp. 147–184.

Y. Almog and B. Helffer.
Global stability of the normal state of superconductors in the presence of a strong electric current
Submitted.

P. Bauman, H. Jadallah, and D. Phillips.
Classical solutions to the time-dependent Ginzburg-Landau equations for a bounded superconducting body in a vacuum.
J. Math. Phys., 46 (2005), pp. 095104, 25.

V. Bonnaillie-Noël and M. Dauge, *Asymptotics for the low-lying eigenstates of the Schrödinger operator with magnetic field near corners*, Ann. Henri Poincaré, 7 (2006), pp. 899–931.

Z. M. Chen, K.-H. Hoffmann, and J. Liang, *On a nonstationary Ginzburg-Landau superconductivity model*, Math. Methods Appl. Sci., 16 (1993), pp. 855–875.

E. B. Davies, *Linear operators and their spectra*, vol. 106 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2007.

R. Dautray and J.L. Lions
Mathematical Analysis and Numerical Methods for Science and Technology.
Vol. 5, Evolution problems I, Springer-Verlag, 1988.

Q. Du, *Global existence and uniqueness of solutions of the time-dependent Ginzburg-Landau model for superconductivity*, Appl. Anal., 53 (1994), pp. 1–17.

Q. Du and P. Gray, *High-kappa limits of the time-dependent Ginzburg-Landau model*, SIAM J. Appl. Math., 56 (1996), pp. 1060–1093.

- K.J. Engel and R. Nagel *One-parameter semigroups for linear evolution equations.* Graduate texts in Mathematics 194, Springer.
- L. C. Evans, *Partial Differential Equations*, AMS, 1st. ed., 1998.
- E. Feireisl and P. Takáv c, *Long-time stabilization of solutions to the Ginzburg-Landau equations of superconductivity*, Monatsh. Math., 133 (2001), pp. 197–221.
- J. Fleckinger-Pellé, H. G. Kaper, and P. Takáv c, *Dynamics of the Ginzburg-Landau equations of superconductivity*, Nonlinear Anal., 32 (1998), pp. 647–665.
- S. Fournais and B. Helffer, *Spectral Methods in Surface Superconductivity*, Birkhäuser, 2009.
- T. Giorgi and D. Philips, *The breakdown of superconductivity due to strong fields for the Ginzburg-Landau model*, SIAM J. Math. Anal., 30 (1999), pp. 341–359.

D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001.
Reprint of the 1998 edition.

V. Girault and P.-A. Raviart, *Finite element approximation of the Navier-Stokes equations*, vol. 749 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, Extended version 1986.

P. Grisvard, *Elliptic problems in nonsmooth domains*, vol. 24 of Monographs and Studies in Mathematics, Pitman (Advanced Publishing Program), Boston, MA, 1985.

P. Grisvard, *Singularities in boundary value problems*. Springer (1992).

B. Helffer and J. Sjöstrand, *From resolvent bounds to semigroup bounds*. Preprint : arXiv:1001.4171v1, (2010).

B. I. Ivlev and N. B. Kopnin, *Electric currents and resistive states in thin superconductors*, Advances in Physics, 33 (1984), pp. 47–114.

T. Kato, *Perturbation theory for linear operators*, Springer, 3rd. ed., 1980.

V.A. Kondratiev, *Boundary Value Problems for elliptic equations in domain with conical or angular points.* Trans. Moscow Math Soc (1967), pp. 227-313 .

K. Lu and X.-B. Pan, *Estimates of the upper critical field for the Ginzburg-Landau equations of superconductivity*, Phys. D, 127 (1999), pp. 73–104.

V.G. Mazya and V.A. Plamenevskii, *L^p estimates of solutions of elliptic boundary problems in domains with edges.* Transactions of the Moscow Mathematical Society (1980) Issue 1, pp. 49-97.

R. MONTGOMERY, *Hearing the zero locus of a magnetic field*, Comm. Math. Phys., 168 (1995), pp. 651–675.

X.-B. PAN AND K.-H. KWEK, *Schrödinger operators with non-degenerately vanishing magnetic fields in bounded domains*, Trans. Amer. Math. Soc., 354 (2002), pp. 4201–4227 (electronic).

M. H. PROTTER AND H. F. WEINBERGER, *Maximum principles in differential equations*, Prentice-Hall, 1967.

J. RUBINSTEIN, P. STERNBERG, AND J. KIM, *On the behavior of a superconducting wire subjected to a constant voltage difference*, SIAM Journal on Applied Mathematics, 70 (2010), pp. 1739–1760.

J. RUBINSTEIN, P. STERNBERG, AND K. ZUMBRUN, *The Resistive State in a Superconducting Wire: Bifurcation from the Normal State*, Archive for Rational Mechanics and Analysis, 195 (2010), pp. 117–158.

J. Sjöstrand,
Resolvent estimates for non-self-adjoint operators via semi-groups.
http://arxiv.org/abs/0906.0094.

I. Tice, *Ginzburg-Landau vortex dynamics driven by an applied boundary current*, Comm. Pure Appl. Math., 63 (2010), pp. 1622–1676.