Behaviour of optimisers for the k^{th} eigenvalue of the Laplacian

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$$\Delta u + \lambda u = 0 \quad \text{in } \Omega$$

(+ a variety of boundary conditions on $\partial\Omega$)

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \to \infty$$

Problem:

Optimise the different λ_j's as a function of the domain

or

Given a frequency f, determine the shape within a class of domains which will support the largest number of modes with frequency below f

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Quick summary of what is *known*

λ_1 and λ_2 are optimised by one ball and two equal balls, respectively

(~130 years)

Optimisers for the Dirichlet problem exist within the class of quasi-open sets

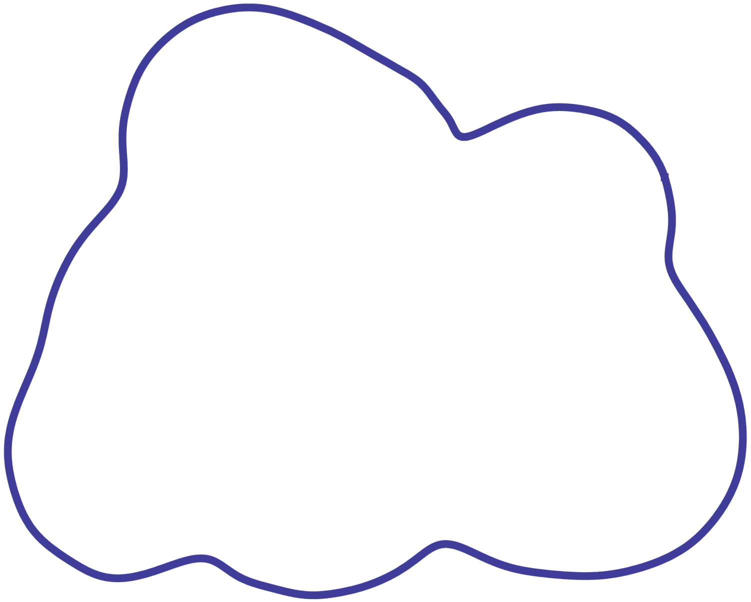
(Bucur and Mazzoleni & Pratelli (2012))

There is no *nice structure* in the mid-frequency range

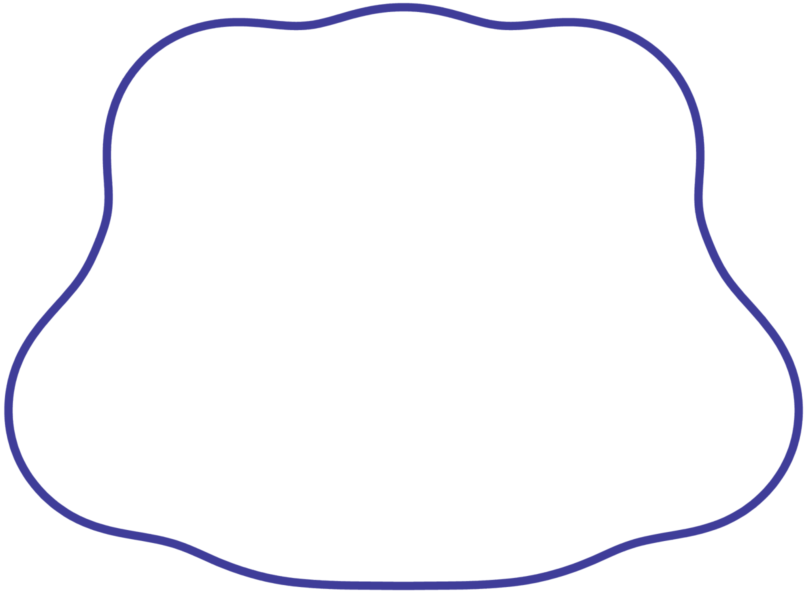
(numerical results within the last 10 years)

optimisers are not described by known functions
there is no (general) symmetry of optimisers

Quick (pictorial) summary of what is *known*

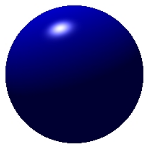
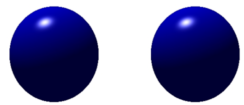
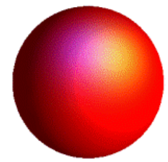
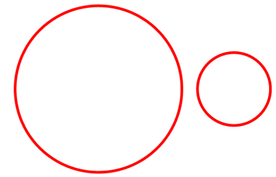
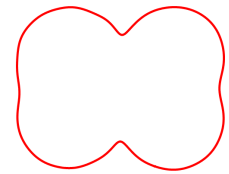
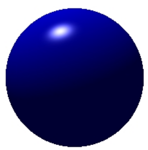
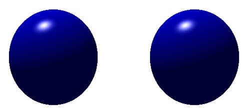
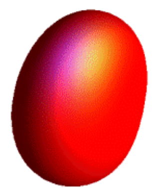
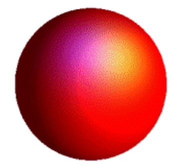
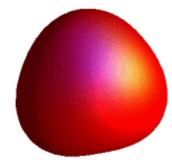
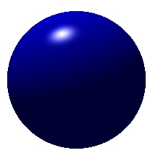
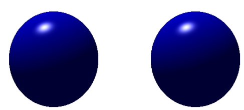
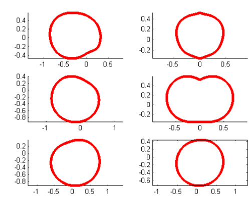
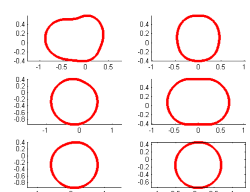
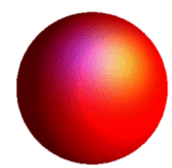


$\lambda_{13} = 186.97$

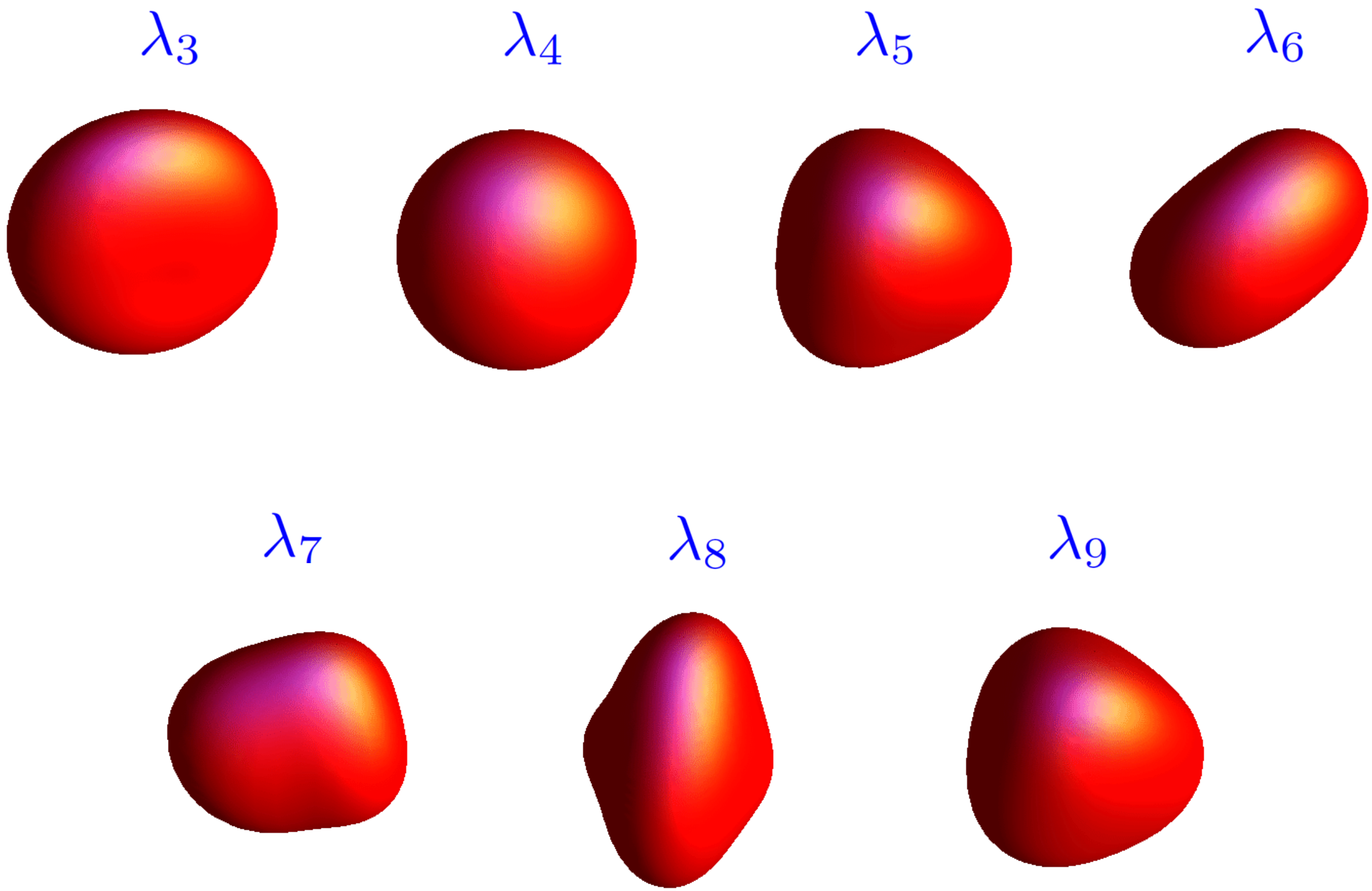


$\lambda^{s}_{13} = 187.92$

Quick (pictorial) summary of what is *known*

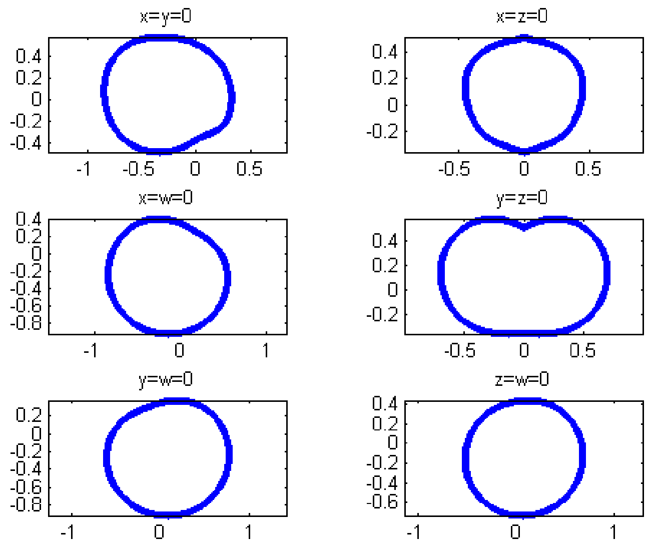
	λ_1	λ_2	λ_3	λ_4	λ_5
2D					
3D					
4D					

3D

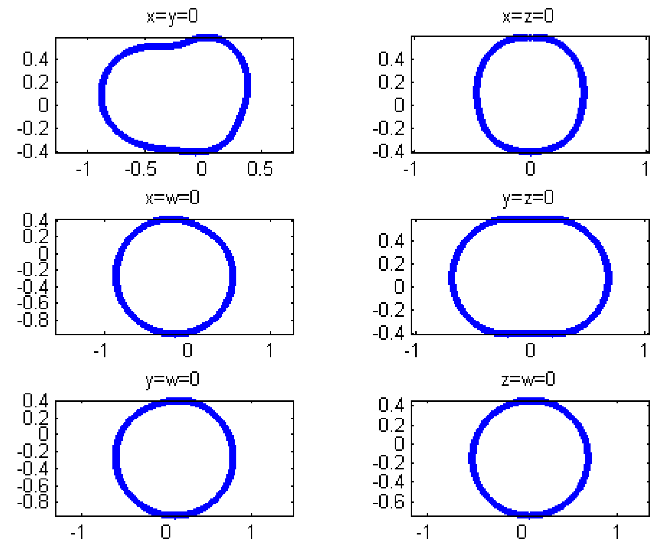


4D

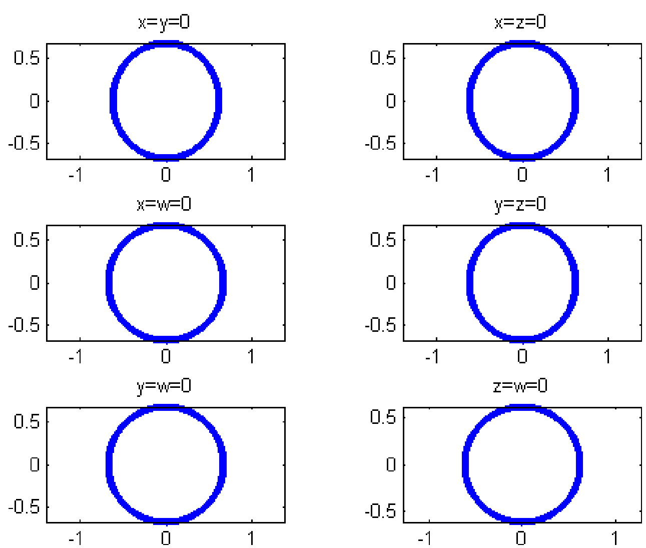
λ_3:



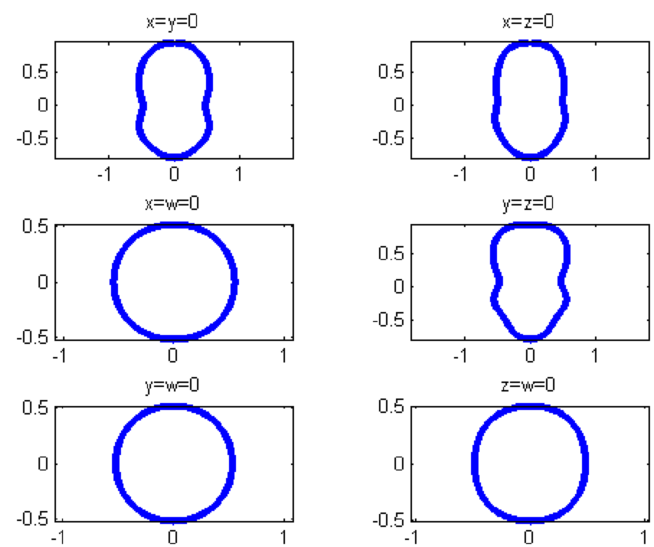
λ_4:



λ_5:



λ_6:



Is it possible to say something about what happens at the *other end* of the spectrum?

$$\lambda_k = \frac{4k\pi}{|\Omega|} + 2\sqrt{\pi}\frac{|\partial\Omega|}{|\Omega|^{3/2}}k^{1/2} + o(k^{1/2}) \quad (k \to \infty)$$

Naively, one might expect that, as k gets larger, minimizing the perimeter and minimizing λ_k would be equivalent, in which case minimizers should approach the ball. However...

...not only is the argument incorrect, but the statement in itself is actually also wrong, in the sense that minimizers do not even need to satisfy the same Weyl asymptotics

Theorem (Antunes, F. and Kennedy 2013) *Given* $V > 0$ *and* $j \geq 1$, *let* B_j *denote the domain of volume* V *consisting of* j *equal balls of radius* $r = (V/j\omega_N)^{1/N}$. *Then, for Robin boundary conditions with any* $\alpha > 0$,

$$\lambda_j^*(V,\alpha) \leq \lambda_j(B_j,\alpha) \leq N\alpha \left(\frac{j\omega_N}{V}\right)^{\frac{1}{N}}.$$

Recall that $\lambda_j(\Omega) = \frac{4\pi^2}{(\omega_N|\Omega|)^{2/N}} j^{2/N} + o(j^{2/N})$ as $j \to \infty$

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The Dirichlet problem satisfies an *extra* condition

$$\lambda_k \geq \frac{2k\pi}{|\Omega|}$$

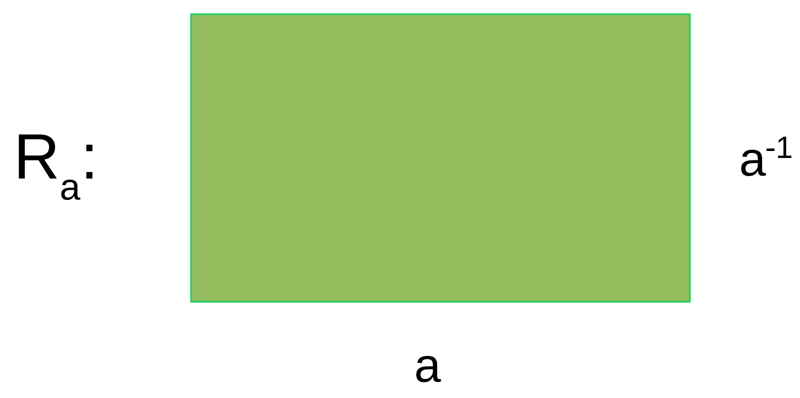
(Berezin, Li & Yau)

or, for tiling domains,

$$\lambda_k \geq \frac{4k\pi}{|\Omega|}$$

(Pólya's inequality)

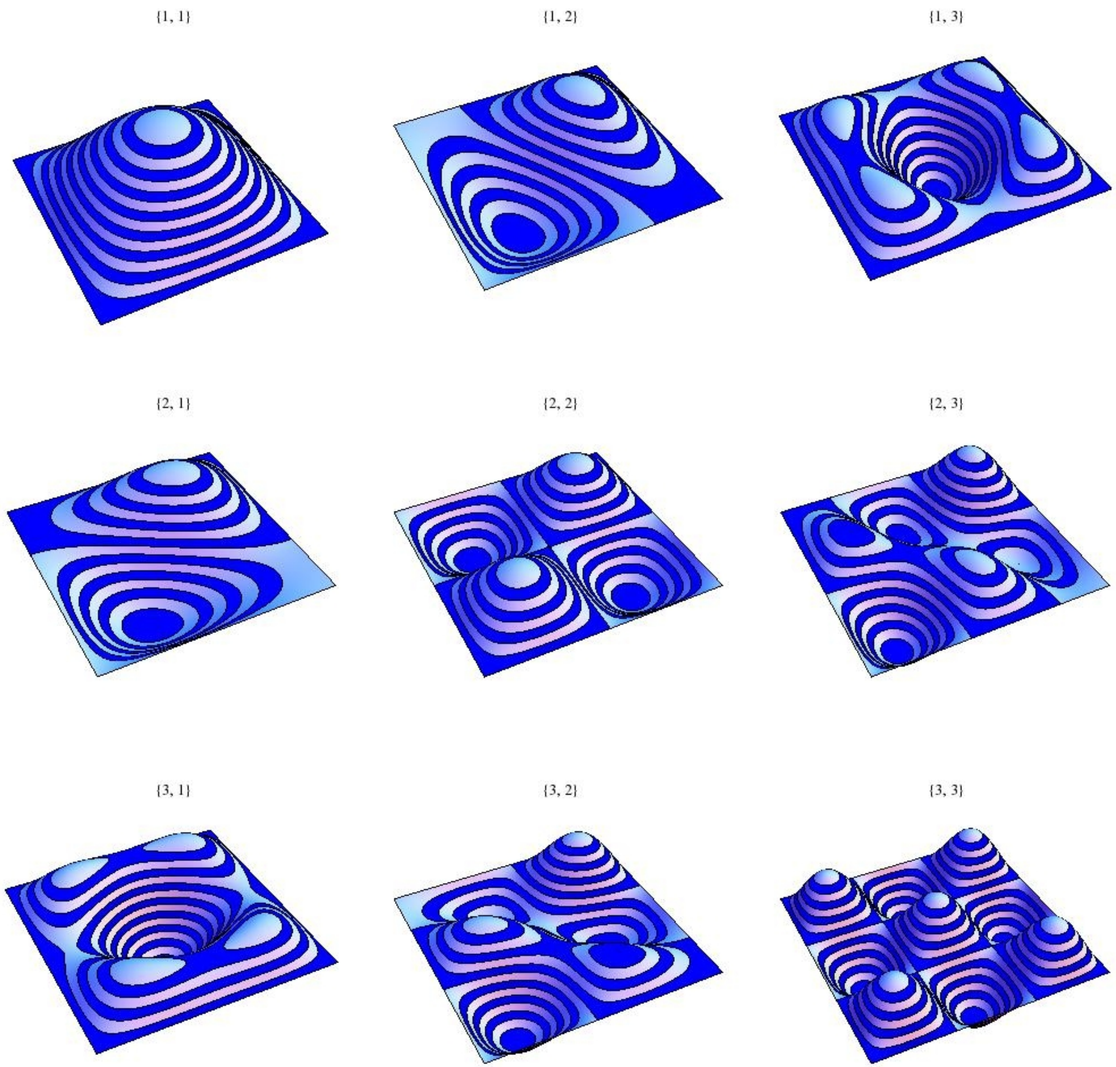
Let's look at a *toy* problem



$$\lambda_k^* = \min_{a \geq 1} \lambda_k(a), \quad k = 1, 2, \ldots .$$

The full consideration of the problem now presenting itself requires the aid of the theory of numbers; but it will be sufficient for the purposes of this work to consider a few of the simpler cases, which arise when the membrane is square. The reader will find fuller information in Riemann's lectures on partial differential equations.

(Rayleigh's *Theory of Sound*)



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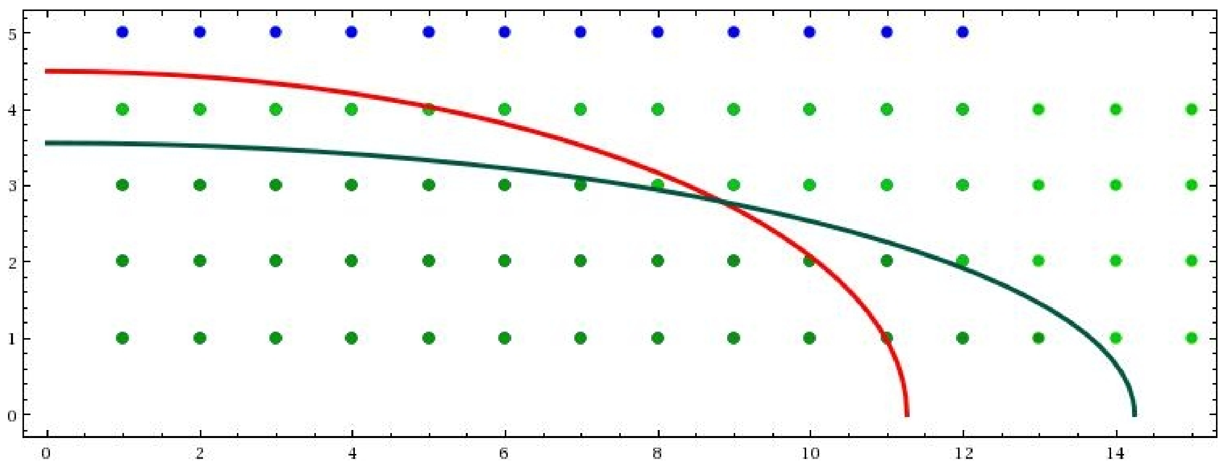
This turns out to be equivalent to the following problem

Among all ellipses centred at the origin with horizontal and vertical axes, determine that with the least area which contains k integer lattice points in the first quadrant (excluding the axes)

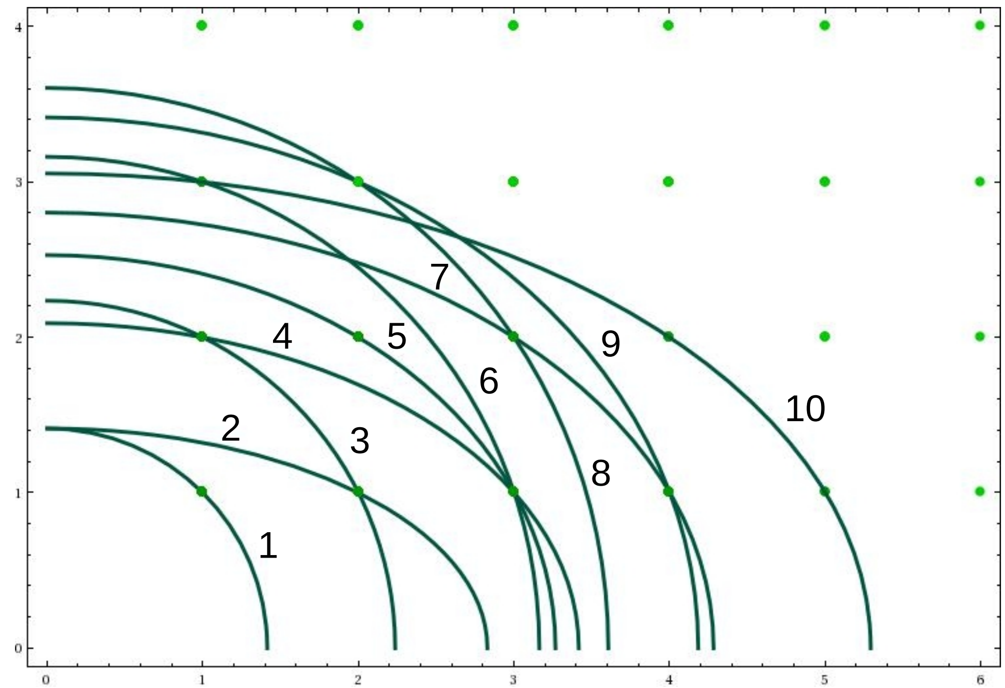


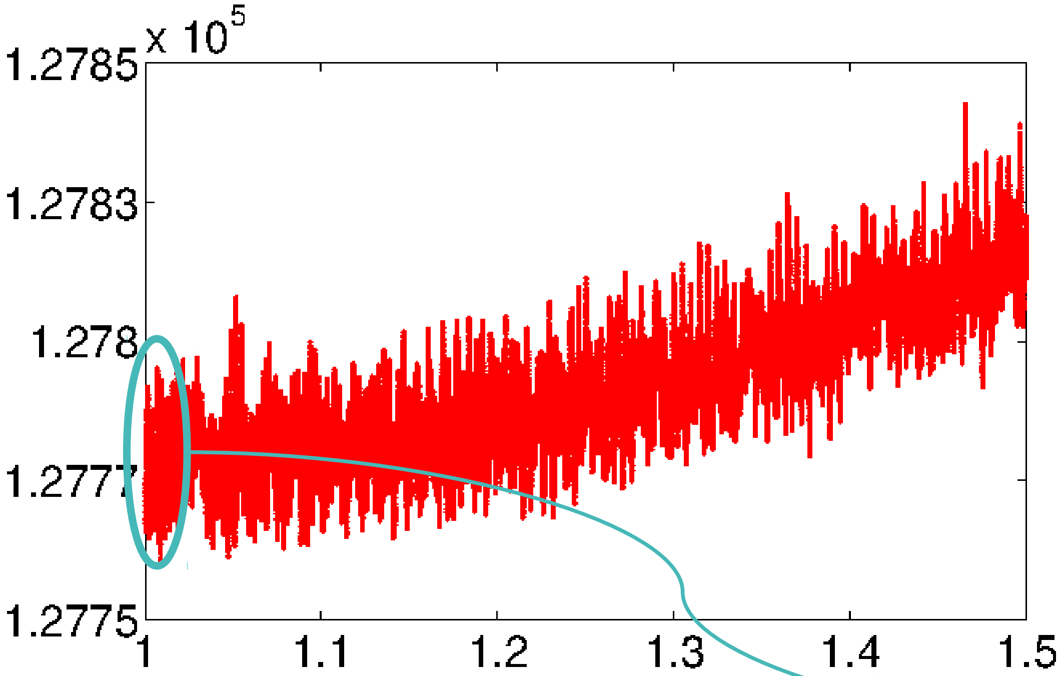
The first 15 optimal rectangles have $(a_k^*)^4$ as follows:

$$1,\quad 4,\quad 1,\quad \frac{8}{3},\quad \frac{5}{3},\quad 1,\quad \frac{7}{3},$$

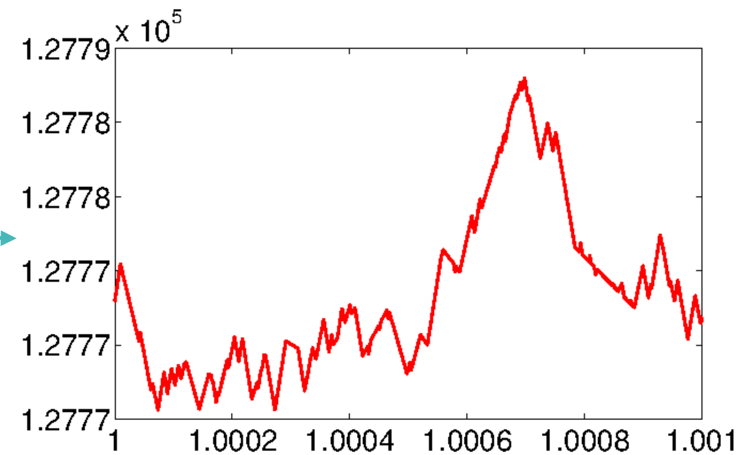
$$1,\quad \frac{3}{2},\quad 3,\quad \frac{21}{8},\quad \frac{5}{4},\quad 1,\quad \frac{27}{8},\quad \frac{9}{5},\quad \ldots$$

Or, in terms of optimal ellipses:





Example of the function
to minimize (λ_{100000})



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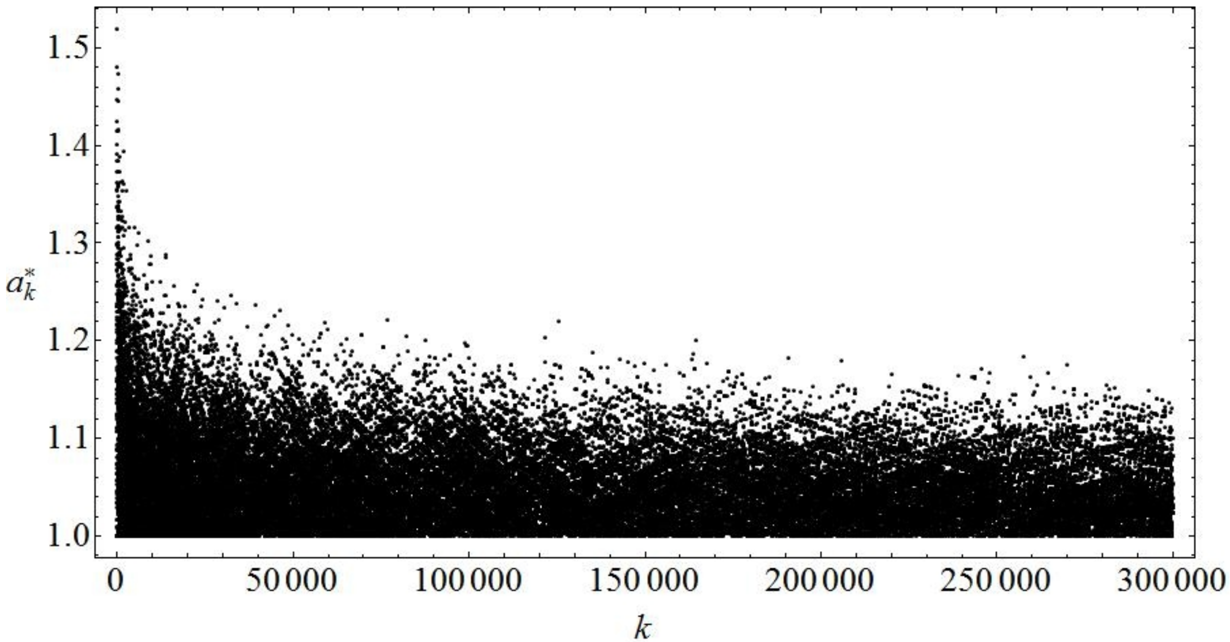
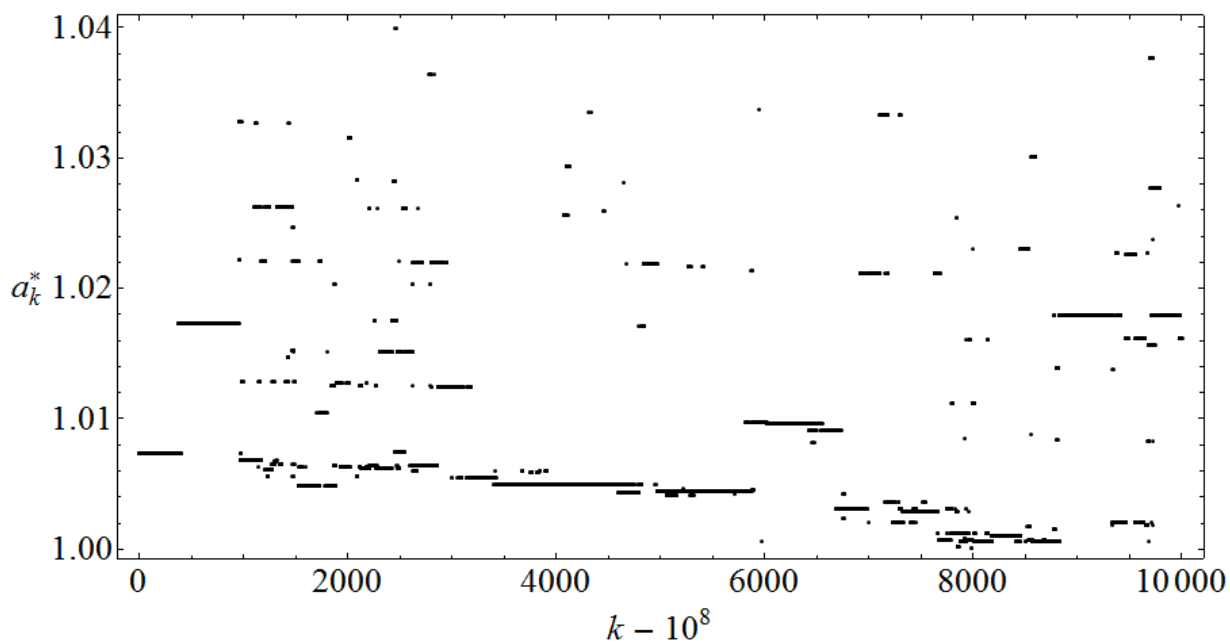

1.5
1.4
1.3
1.2
1.1
1.0
a_k^*
0
50 000
100 000
150 000
200 000
250 000
300 000
k


1.04
1.03
1.02
1.01
1.00
a_k^*
0
2000
4000
6000
8000
10 000
$k - 10^8$

Theorem (**Antunes and F. 2013**)

$$\lim_{k \to \infty} a_k^* = 1,$$

that is, the asymptotic optimal domain is the square.

Remark:

Known results for the lattice problem

$$N_0(r) \leq \pi r^2 + C r^{131/208} \log^{18637/8320}(r)$$

(Huxley 2003)

$$N_0(r) \leq \pi r^2 + \frac{17}{2} r^{2/3} + \left(3a^{3/2} + \frac{700}{a^{3/2}}\right) r^{1/2} + 11$$

(Krätzel and Nowak 2004)

(idea of) **Proof:**

1. Improve Pólya's lower bound for eigenvalues of rectangles

2. Prove boundedness of the sequence of optimal sides

3. Use results from the lattice problem to show convergence to the square

Theorem *For all rectangles R_a and all k we have*

$$\lambda_k \geq 4\pi k + 2a\lambda_k^{1/2} - \frac{4\sqrt{2\pi}}{3\sqrt{3}} a^{3/2} \lambda_k^{1/4}.$$

Lemma

$$\limsup_{k\to+\infty} a_k^* \leq \frac{6\sqrt{3}}{3\sqrt{3}-2\sqrt{2}} \approx 4.38915.$$

Remark: if one uses the best known result for ellipses

$$N_0(r) \leq \pi r^2 + \frac{17}{2} r^{2/3} + \left(3a^{3/2} + \frac{700}{a^{3/2}}\right) r^{1/2} + 11$$

$$\downarrow$$

$$\begin{aligned} \lambda_k \;\; \geq \;\; & 4k\pi + 2\left(a + \frac{1}{a}\right)\lambda_k^{1/2} - \frac{17}{2}\pi^{1/3}\lambda_k^{1/3} \\ & -\sqrt{\pi}\left(3a^{3/2} + \frac{700}{a^{3/2}}\right)\lambda_k^{1/4} - 47\pi \end{aligned}$$

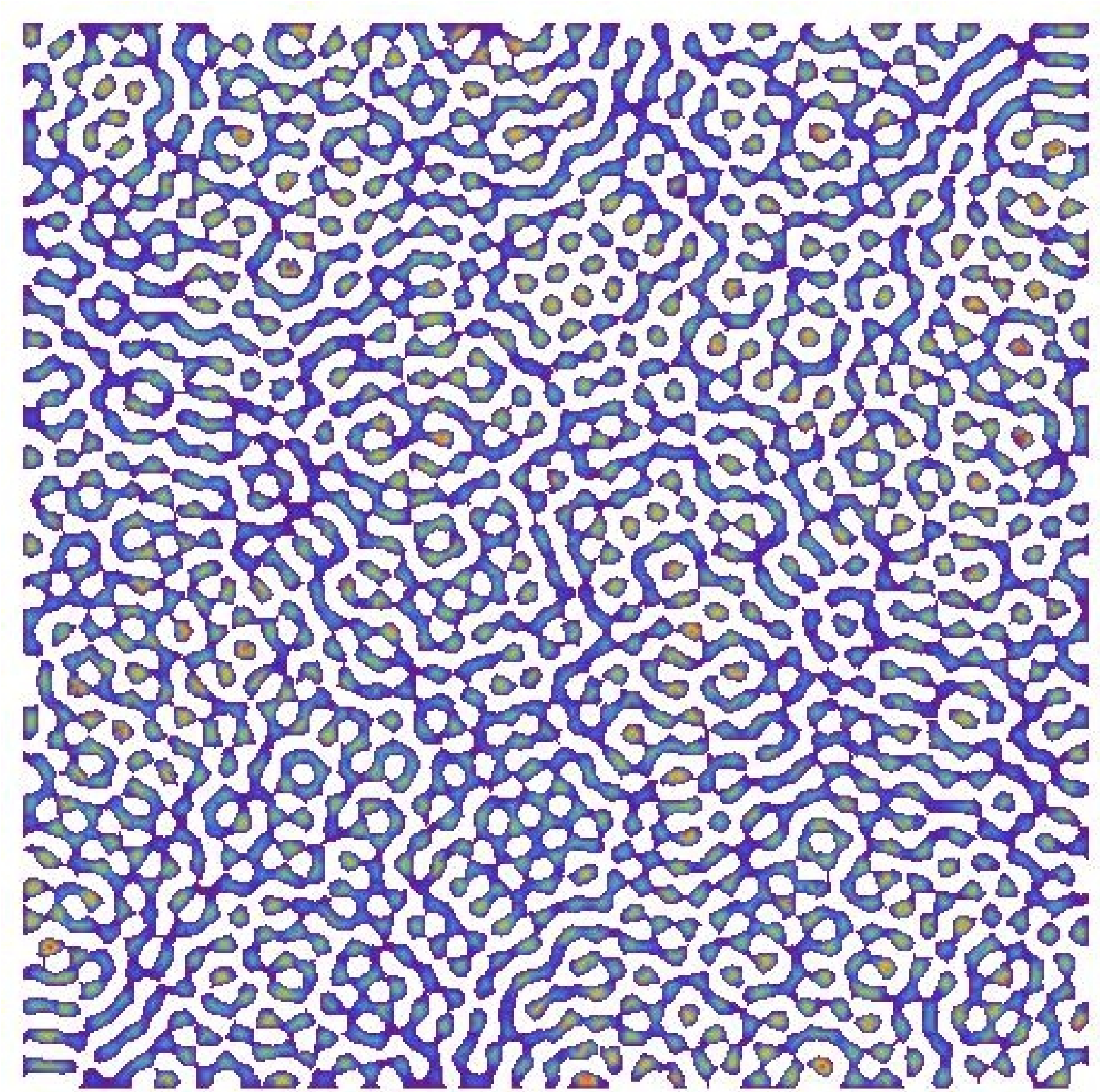
$$N_0(r) \leq \pi r_1 r_2 + C r^\theta \qquad (\exists\, \theta \in (1/2, 1),\ C > 0)$$

$$\downarrow$$

$$\lambda_k \geq 4k\pi + 2\left(a + \frac{1}{a}\right)\lambda_k^{1/2} - C'\lambda_k^{\theta/2} - 3\pi$$

$\lambda^{*}_{4276} = 5525\pi^2$

(mult: 12)



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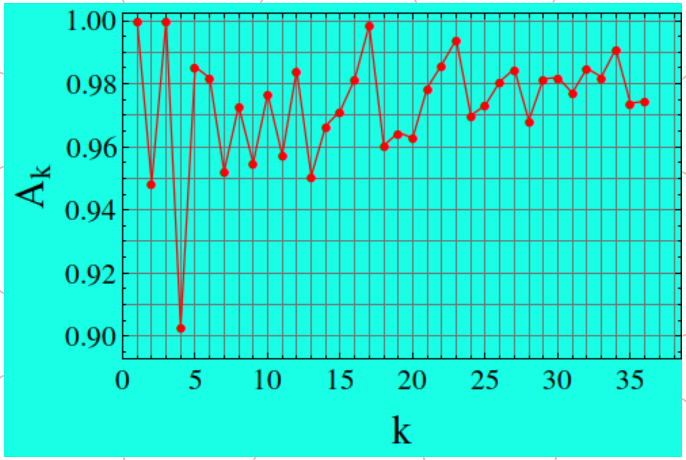
Theorem (**Bucur and F. 2013**) *The sequence* Ω_k^* *of optimal planar domains with fixed perimeter converges to* $\Omega_\infty^* = D$ *as* k *goes to infinity, where* D *denotes the disk with the same perimeter.*



Theorem *Let $\mathcal{P}_n$ denote the family of $n-$sided planar polygons and consider the minimisation problem*

$$\lambda_k^* = \min\{\lambda_k(P) : P \in \mathcal{P}_n,\ |\partial P| = \alpha\} \qquad (1)$$

Then the sequence P_k^* of optimal $n-$polygons with fixed perimeter converges to the regular $n-$polygon with the same perimeter as k goes to infinity.