Counting intersections of nodal lines with curves on real analytic surfaces

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Spectral Theory of Laplace and Schroedinger Operators
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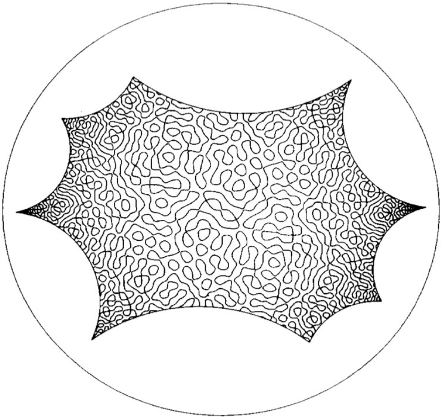
$$-\Delta_g \varphi_\lambda = \lambda^2 \varphi_\lambda.$$

- The nodal set of φ_λ is by definition

$$\mathcal{N}_{\varphi_\lambda} = \{x \in M : \varphi_\lambda(x) = 0\}.$$

$\mathcal{N}_{\varphi_\lambda}$ is the **least** likely place for a quantum particle in the state φ_λ to be.

Nodal sets



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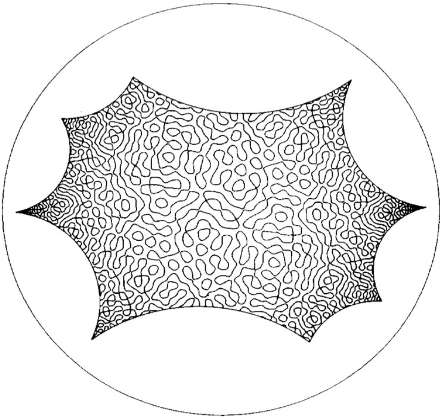
Length [Donelly-Fefferman '88]: length$(\mathcal{N}_{\varphi_\lambda}) \asymp \lambda$



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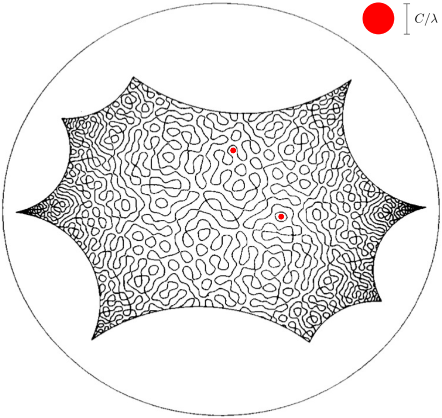
Inner radius [Brüning '78],[Mangoubi '06]: inrad$(\textit{nodal domain of } \varphi_\lambda) \asymp \lambda^{-1}$



The Problem

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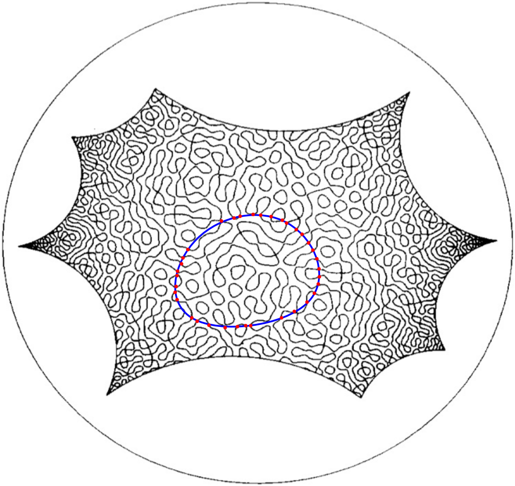
- For H real analytic curve, find upper bounds for $\#(\mathcal{N}_{\varphi_\lambda} \cap H)$.



“Bad” curves on the **Torus**

On $M = \mathbb{T}^2$, the eigenfunctions

$$\varphi_{n,m}(x,y) = \sin(2\pi nx)\sin(2\pi my)$$

vanish on $H = \{y = 0\}$ and on $H = \{x = 0\}$.

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A more general result holds:

Theorem [Bourgain-Rudnick, 2010].

H is a segment of a closed geodesic $\quad\Leftrightarrow\quad \exists\, \{\varphi_{\lambda_{j_k}}\}_k$ with $\varphi_{\lambda_{j_k}}|_H = 0$.

Good curves

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- **Definition.** A curve H is said to be **good** if for some constants $C_0 > 0$, $\lambda_0 > 0$
$$\|\varphi_\lambda\|_{L^2(H)} \geq e^{-C_0\lambda} \qquad \text{for all } \lambda > \lambda_0.$$

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- **Example.** The domain boundary $H = \partial\Omega$ for $\Omega \subset \mathbb{R}^2$ is always good (Neumann boundary conditions).

- The goodness condition is likely to be generically satisfied **BUT** for general curves the goodness condition is not easy to verify for all eigenfunctions.

Positive results known

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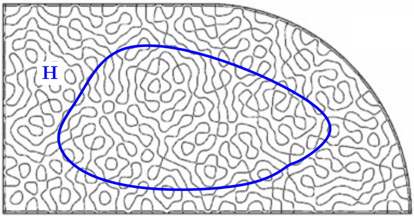
Theorem [Toth - Zelditch, 2009]
Let $\Omega \subset \mathbb{R}^2$ be an analytic, bounded planar domain. Let $H \subset \text{int}(\Omega)$ be a real analytic good curve. For all Neumann eigenfunctions

$$\#(\mathcal{N}_{\varphi_\lambda} \cap H) \leq O_{H,\Omega}(\lambda).$$



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Theorem [Burgain-Rudnick, 2010].

Let $M = \mathbb{T}^2$ and $H \subset M$ have strictly positive curvature.

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Theorem [Jung, 2011].

Let $M =$ compact hyperbolic surface and H=geodesic circle.

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Main result: Compact surfaces

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Then, for all $\lambda \geq \lambda_0$,
$$\#(\mathcal{N}_{\varphi_\lambda} \cap H) \leq O_{M,H}(\lambda).$$

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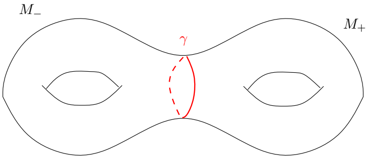
$$M = M_- \cup M_+, \qquad \gamma = \partial M_+.$$



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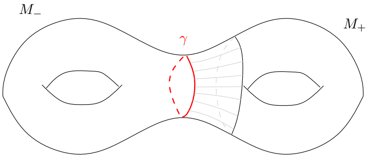

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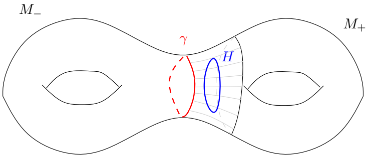

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Let $(\varphi_{\lambda_{j_k}})_{j=1}^{\infty}$ be a quantum ergodic sequence of Laplace eigenfunctions that are even (odd) with respect to the involution.

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Remark. The result holds for ALL eigenfunctions on QUE surfaces with isometric involution (e.g arithmetic surfaces with isometric involutions [Lindenstrauss]).

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  Example: squared Riemann distance $r^2(x_1, x_2)$ is analytic close to the diagonal so it extends to $r^2_{\mathbb{C}}(z_1, z_2)$.

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- There is a maximal Grauert tube radius ε_{max}. For $\varepsilon \leq \varepsilon_{max}$

$$\varphi_\lambda^{\mathbb{C}} : M_\varepsilon^{\mathbb{C}} \to \mathbb{C}$$

  is holomorphic.

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Theorem [Lin (1991)] There exists a universal $r \in (0,1)$ for which

$$\#\{\mathcal{N}_v \cap B_r\} \leq 2F(v)$$

for all $v \in C^\omega(\bar{B}_1)$.

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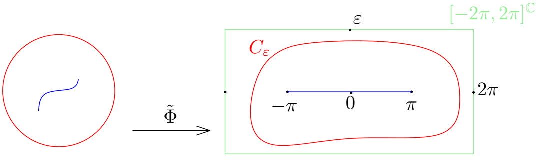
- By Riemman mapping theorem we may think of C_ε as the unit disc B_1.



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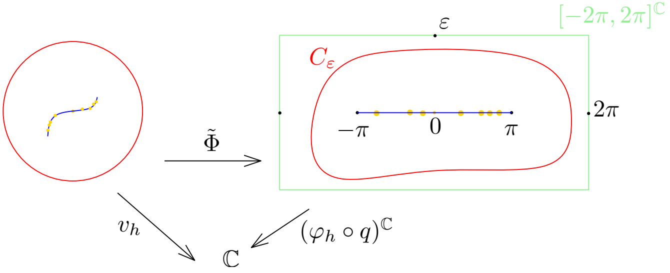

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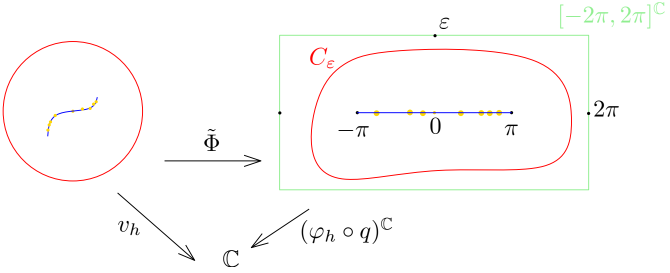

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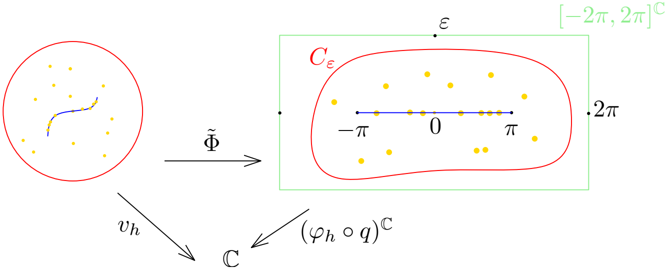

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- Applying the change of variables $z \mapsto \tilde{\Phi}(z)$,

$$\#(\mathcal{N}_{\varphi_h} \cap H) \leq C\frac{\|\partial_T (\varphi_h \circ q)^{\mathbb{C}}\|_{L^2(\partial C_\varepsilon)}}{\|(\varphi_h \circ q)^{\mathbb{C}}\|_{L^2(\partial C_\varepsilon)}}.$$

Main Theorem: H good $\Rightarrow \#(\mathcal{N}_{\varphi_\lambda} \cap H) = O(\lambda)$

Let $\chi_R \in C_0^\infty(T^*\partial C_\varepsilon)$ with

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$$\leq \underbrace{C \frac{\|Op_h(\chi_R)\partial_T (\varphi_h \circ q)^{\mathbb{C}}\|_{L^2(\partial C_\varepsilon)}}{\|(\varphi_h \circ q)^{\mathbb{C}}\|_{L^2(\partial C_\varepsilon)}}}_{O(h^{-1})=O(\lambda)} + \underbrace{C \frac{\|(1 - Op_h(\chi_R))\partial_T (\varphi_h \circ q)^{\mathbb{C}}\|_{L^2(\partial C_\varepsilon)}}{\|(\varphi_h \circ q)^{\mathbb{C}}\|_{L^2(\partial C_\varepsilon)}}}_{O(h^{-1}e^{-C/h})}.$$

- Follows from L^2-boundedness of $Op_h(\chi_R)h\partial_T$

Main Theorem: H good $\Rightarrow \#(\mathcal{N}_{\varphi_\lambda} \cap H) = O(\lambda)$

Let $\chi_R \in C_0^\infty(T^*\partial C_\varepsilon)$ with

$$\chi_R(s,\sigma) = \begin{cases} 1 & |\sigma| \leq R \\ 0 & |\sigma| \geq R+1. \end{cases}$$

And so

$$\#(\mathcal{N}_{\varphi_h} \cap H) \leq C \frac{\|\partial_T (\varphi_h \circ q)^{\mathbb{C}}\|_{L^2(\partial C_\varepsilon)}}{\|(\varphi_h \circ q)^{\mathbb{C}}\|_{L^2(\partial C_\varepsilon)}}$$

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- We use the complexified Heat kernel to reproduce the eigenfunctions

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$$\leq \underbrace{C \frac{\|Op_h(\chi_{_R})\partial_{_T}(\varphi_h \circ q)^{\mathbb{C}}\|_{L^2(\partial C_\varepsilon)}}{\|(\varphi_h \circ q)^{\mathbb{C}}\|_{L^2(\partial C_\varepsilon)}}}_{O(h^{-1})=O(\lambda)} + \underbrace{C \frac{\|(1 - Op_h(\chi_{_R}))\partial_{_T}(\varphi_h \circ q)^{\mathbb{C}}\|_{L^2(\partial C_\varepsilon)}}{\|(\varphi_h \circ q)^{\mathbb{C}}\|_{L^2(\partial C_\varepsilon)}}}_{O(h^{-1}e^{-C/h})}.$$

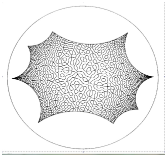
- Follows from L^2-boundedness of $Op_h(\chi_{_R})h\partial_{_T}$

- We use the complexified Heat kernel to reproduce the eigenfunctions

- We use contour deformation of ∂C_ε to make the phase of the FIO be positive

Picture credits

R. Aurich and F. Steiner. "Statistical properties of highly excited quantum eigenstates of a strongly chaotic system." Physica D: Nonlinear Phenomena 64.1 (1993): 185-214.



M. Berry and H. Ishio. "Nodal-line densities of chaotic quantum billiard modes satisfying mixed boundary conditions." Journal of Physics A: Mathematical and General 38.29 (2005): L513.